GENERALIZED L_2-LAPLACIANS

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1. Given a function $F(x, y)$ in L_2 on the plane, we shall say that $F(x, y)$ has $f(x, y)$ as a generalized L_2-Laplacian if

$$8\left[\pi^{-1}\int_{D_2(0,1)} F(x+tu, y+tv)dudv - F(x, y)\right]\Big/ t^2 \to f(x, y) \text{ in the } L_2\text{-norm}$$

where $D_2(0, 1)$ is the disc with radius 1 and center at the origin. It will be shown here that a necessary and sufficient condition that $F(x, y)$ has $f(x, y)$ as a generalized L_2-Laplacian is that $F(x, y)$ be locally an L_2-potential of $f(x, y)$, that is given any disc $D_2(0, R)$ there is a harmonic function $h_R(x, y)$ such that for almost all (x, y) in $D_2(0, R)$

$$F(x, y) = -(2\pi)^{-1}\int_{D_2(0,R)} \log\left[(x-u)^2 + (y-v)^2\right]^{-1/2} f(u, v)dudv + h_R(x, y)$$

where $f(x, y)$ is in L_2 on the plane.

This result will be seen on one hand to be the two-dimensional analogue of the theorems given in [1, pp. 130–131] on the connection between locally absolutely continuous functions and the L_2-derivative in one dimension and on the other hand to be the L_2-analogue of the theorems obtained in [6] for pointwise convergence of the line integral analogue of the above generalized Laplace operator. It will be clear from what follows that a similar theory can be developed in the L_1-case and also in the L_p-case with p between 1 and 2.

2. We shall operate in n-dimensional Euclidean space E_n, $n \geqq 2$, and shall use vectorial notation. Thus,

$$x = (x_1, \cdots, x_n), \quad y = (y_1, \cdots, y_n), \quad (x, y) = x_1y_1 + \cdots + x_ny_n,$$
$$\alpha x + \beta y = (\alpha x_1 + \beta y_1 + \cdots + \alpha x_n + \beta y_n), \qquad (x, x)^{1/2} = |x|.$$

The open n-dimensional sphere with center x_0 and radius r will be designated by $D_n(x_0, r)$. Given $F(x)$ integrable on $D_n(x_0, r)$, we shall designate the mean value of F in this sphere by $A(F, x_0, r)$. Thus let-

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ting $2\pi^{n/2}/n\Gamma(n/2)=\Omega_n$, which is the volume of the unit n-dimensional sphere, we have that

$$A(F, x_0, r) = (\Omega_n r^n)^{-1} \int_{D_n(x_0,r)} F(x)dx.$$

We further set $\nabla(F, x_0, r)=A(F, x_0, r)-F(x_0)$ and say that $F(x)$ has $f(x)$ as a generalized L_2-Laplacian, designated by $L_2\Delta F=f$, if

$$\|2(n+2)\nabla(F, x, r)/r^2 - f(x)\|_{L_2} \to 0 \qquad \text{as } r \to 0,$$

that is if

$$\lim_{r\to 0} \left\{ \int_{E_n} |\, 2(n+2)\nabla(F, x, r)/r^2 - f(x)\,|^2 dx \right\}^{1/2} = 0.$$

$F(x)$ will be said to be a local L_2-potential of $f(x)$ if for every $r>0$ there is a function $h_r(x)$, harmonic in $D_n(0, r)$, such that

$$F(x) = -\omega_n^{-1} \int_{D_n(0,r)} \Phi_n(x-y)f(y)dy + h_r(x)$$

for almost all x in $D_n(0, r)$, where $\Phi_n(x)=|x|^{-(n-2)}(n-2)^{-1}$ if $n\geqq 3$ and $=\log|x|^{-1}$ if $n=2$, ω_n is the $(n-1)$-dimensional volume of the surface of the unit sphere, and where $f(x)$ is in L_2 on E_n.

$G(u)$ will be said to be the Fourier transform of $F(x)$ if

$$G(u) = (2\pi)^{-n} \operatorname*{l.i.m.}_{R\to\infty} \int_{D_n(0,R)} e^{-i(x,u)}F(x)dx.$$

3. We shall prove the following theorem:

THEOREM 1. *A necessary and sufficient condition that $F(x)$ in L_2 on E_n has $f(x)$ as a generalized L_2-Laplacian is that $F(x)$ be a local L_2-potential of $f(x)$.*

Before proving this theorem, we point out that if $F(x)$ has $f(x)$ as a generalized L_2-Laplacian, then by the above theorem and [2], [3], and [4], $F(x)$ has an ordinary Laplacian almost everywhere equal to $f(x)$. That the converse of this statement does not hold can be seen in the following manner. Take E to be a closed planar set of measure zero and positive capacity contained in the interior of the disc $D_2(0, 1/2)$. Let $F(x)$ be the equilibrium potential of unit mass distributed on E. Then $F(x)$ is harmonic everywhere on the complement of E but not in the whole disc $D_2(0, 1/2)$. Let $\lambda(x)$ be the localizing function of class $C^{(\infty)}$ for the discs $D_2(0, 1/2)$ and $D_2(0, 1)$, that is, $\lambda(x)=1$ for x in $D_2(0, 1/2)$ and $\lambda(x)=0$ for x in the complement of

$D_2(0, 1)$. Then $\lambda(x)F(x)$ is in L_2 on the plane and $\Delta[\lambda(x)F(x)]$ exists almost everywhere and is also in L_2 on the plane. But $\Delta(\lambda F)$ is not the generalized L_2-Laplacian of λF. For if it were by the theorem to be proved in this paper, F would be harmonic in the whole disc $D_2(0, 1/2)$, which it is not.

4. In order to prove the necessary condition of the theorem, we need the following lemma:

LEMMA. *Let $F(x)$ and $f(x)$ be in L_2 on E_n and suppose that $G(u)$ and $g(u)$ are their respective Fourier transforms. Then if for almost all u, $g(u) = -|u|^2 G(u)$, $F(x)$ is a local L_2-potential of $f(x)$.*

To prove the lemma, set $F_R(x) = \int_{E_n} e^{-|u|/R} e^{i(x,u)} G(u)du$ and $f_R(x) = \int_{E_n} e^{-|u|/R} e^{i(x,u)} g(u)du$. Then $\Delta F_R(u) = f_R(u)$, and consequently for x in $D_n(0, T)$,

$$F_R(x) = -\omega_n^{-1} \int_{D_n(0,T)} \Phi_n(x-y) f_R(y)dy + h_{R,T}(x) \tag{1}$$

where $h_{R,T}(x)$ is harmonic in $D_n(0, T)$. Since $F_R(x)$ and $f_R(x)$ are in $C^{(\infty)}$ in E_n, we also have that $h_{R,T}(x)$ is in L_1 on $D_n(0, T)$. Next we observe that $F_R(x)$ tends to $F(x)$ in the L_1-norm over $D_n(0, T)$, and that by Fubini's theorem and Schwarz's inequality,

$$\int_{D_n(0,T)} dx \int_{D_n(0,T)} |\Phi_n(x-y)|\,|f_R(y) - f(y)|\,dy$$
$$\leqq K_T \|f_R - f\|_{L_2} \to 0 \qquad \text{as } R \to \infty$$

where K_T is a constant depending on T. But then from (1) we see that $h_{R,T}(x)$ tends in L_1-norm over $D_n(0, T)$ to a function $h_T(x)$ which by [5, p. 20] can be taken to be harmonic in $D_n(0, T)$. We consequently conclude from (1) that for almost all x in $D_n(0, T)$,

$$F(x) = -\omega_n^{-1} \int_{D_n(0,T)} \Phi_n(x-y) f(y)dy + h_T(x),$$

which fact proves the lemma.

5. To prove the necessary condition of the theorem, let $G(u)$ and $g(u)$ be the respective Fourier transforms of $F(x)$ and $f(x)$. Then since for fixed t, $A(F, x, t)$ is in L_2 on E_n, we have that $\nabla(F, x, t) = A(F, x, t) - F(x)$ is in L_2 on E_n. This in conjunction with the fact that for fixed u, $A(e^{i(x,u)}, x_0, t) = 2^{\mu+1}\Gamma(\mu+2)(|u|t)^{-(\mu+1)} J_{\mu+1}(|u|t) e^{i(x_0,u)}$ where $\mu = (n-2)/2$ and $J_{\mu+1}(t)$ is the Bessel function of the first kind

of order $\mu+1$, tells us that the Fourier transform of $2(n+2)\nabla(F, x, t)$ is $\eta(|u|t)G(u)$ where

$$\eta(t) = -4(\mu+2)[1 - 2^{\mu+1}\Gamma(\mu+2)t^{-(\mu+1)}J_{\mu+1}(t)].$$

By assumption we have that $\|2(n+2)\nabla(F, x, t)/t^2 - f(x)\|L_2 \to 0$ as $t \to 0$; so by the Plancherel theorem, we obtain that $\|\eta(|u|t)G(u)/t^2 - g(u)\|_{L_2} \to 0$. Therefore there exists a sequence $\{t_j\}$ with $t_j \to 0$ as $j \to \infty$ such that for almost all u, $\eta(|u|t_j)G(u)/t_j^2 \to g(u)$. But $\eta(|u|t_j)/t_j^2 \to -|u|^2$, and we conclude that $-|u|^2G(u) = g(u)$ for almost all u. The lemma then tells us that $F(x)$ is a local L_2-potential of $f(x)$ and the necessary part of the theorem is proved.

6. To prove the sufficiency part of the theorem, let t less than 1 be given and choose R large. Then by assumption for almost all x in $D_n(0, R/2)$,

$$F(x) = -\omega_n^{-1}\int_{D_n(0,R)} \Phi_n(x-y)f(y)dy + h_R(x) \tag{2}$$

where $h_R(x)$ is harmonic in $D_n(0, R)$. Let x_0 be a point in $D_n(0, R/2)$ for which (2) holds. Then

$$\nabla(F, x_0, t) = -\omega_n^{-1}\int_{D_n(0,R)} f(y)\nabla(\Phi_n(x-y), x_0, t)dy.$$

But from the fact that for fixed y,

$$\begin{aligned} A(\Phi_n(x-y), x_0, t) &= \Phi_n(x_0 - y) && \text{if } |x_0 - y| > t \\ &= nt^{-n}2^{-1}[t^2(n-2)^{-1} - |x_0 - y|^2 n^{-1}] \\ &\qquad\qquad \text{if } |x_0 - y| \leqq t \text{ and } n \geqq 3 \\ &= 2^{-1}[1 - 2\log t - |x_0 - y|^2 t^{-2}] \\ &\qquad\qquad \text{if } |x_0 - y| \leqq t \text{ and } n = 2, \end{aligned}$$

we obtain that

$$\nabla(F, x_0, t) = -\omega_n^{-1}\int_{D_n(x_0,t)} f(y)\nabla(\Phi_n(x-y), x_0, t)dy.$$

We conclude that for almost all x in E_n,

$$\begin{aligned} &[2(n+2)\nabla(F, x, t)/t^2 - f(x)] \\ &\qquad = -\omega_n^{-1}2(n+2)\int_{D_n(0,t)} [f(x+y) - f(x)]\psi_n(t, y)dy \end{aligned} \tag{3}$$

where

$$\psi_n(t, y) = \{nt^{-n}2^{-1}[t^2(n-2)^{-1} - |y|^{2}n^{-1}] - |y|^{2-n}(n-2)^{-1}\}/t^2 \quad \text{if } n \geqq 3,$$

$$= \{2^{-1}[1 - |y|^2t^{-2}] - \log t\,|y|^{-1}\}/t^2 \quad \text{if } n = 2,$$

$$\int_{D_n(0,t)} \psi_n(t, y)dy = -\omega_n[2(n+2)]^{-1}$$

and

$$\int_{D_n(0,t)} |\psi_n(t, y)|\, dy \leqq K_n \text{ where } K_n \text{ is a constant independent of } t. \tag{4}$$

From (3) and (4) and the Schwarz inequality, we obtain that for almost all x in E_n,

$$\begin{aligned} &|2(n+2)\nabla(F, x, t)/t^2 - f(x)|^2 \\ &\qquad \leqq K_n' \int_{D_n(0,t)} |f(x+y) - f(x)|^2 |\psi_n(t, y)| dy \end{aligned} \tag{5}$$

where K_n' is another constant independent of t. From (4), (5), and Fubini's theorem, we conclude that

$$\begin{aligned} \|2(n+2)\nabla(F, x, t)/t^2 - f(x)\|_{L_2} &\leqq K_n'' \max_{0<|y|\leqq t} \|f(x+y) - f(x)\|_{L_2} \\ &= o(1) \text{ as } t \to 0, \end{aligned}$$

which fact is the desired result.

7. Extending the definition for the generalized L_2-Laplacian and local L_2-potential in an obvious manner to L_p-spaces for $1 \leqq p \leqq 2$, using the Hölder inequality or obvious facts about functions in L_1 where we have previously used the Schwartz inequality and using the Titchmarsh-Hausdorff-Young theorem where we have previously used the Plancherel theorem, we obtain the following extension of Theorem 1.

THEOREM 2. *Let $1 \leqq p \leqq 2$. Then a necessary and sufficient condition that $F(x)$ in L_p on E_n has $f(x)$ as a generalized L_p-Laplacian is that $F(x)$ be a local L_p-potential of $f(x)$.*

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