ON A CLASS OF SEMIGROUPS ON $E_n$

PAUL S. MOSTERT AND ALLEN L. SHIELDS

Very little is known about continuous, associative multiplications with identity on a manifold unless the manifold is compact (in which case it must be a Lie group [1]). In this, and in forthcoming papers, the authors will study this problem, particularly when the manifold is $E_n$ ($n$-dimensional Euclidean space), and with other restrictions. Here, we classify all (topological) semigroups on the half line $[0, \infty)$ in which 0 and 1 play their natural roles of zero and identity, and use this as an aid in classifying semigroups $S$ with identity on $E_n$, $n > 1$, such that some $(n-1)$-dimensional compact connected submanifold is a subsemigroup containing the identity of $S$. It turns out that only $E_2$ and $E_4$ will admit such a situation. In fact, we prove the following two theorems (see §1 for definitions):

**Theorem A.** Let $S$ be the half-line $[0, \infty)$. Suppose $S$ is a semigroup with zero at 0 and identity at 1. Then

(i) if $S$ contains no other idempotents, its multiplication is the ordinary multiplication of real numbers on $[0, \infty)$;

(ii) if $S$ contains an idempotent different from 0 and 1, then it contains a largest (in the sense of the regular order of real numbers) such idempotent $e$. Moreover, $e < 1$, $[e, \infty)$ is a subsemigroup isomorphic to $[0, \infty)$ under the usual multiplication of real numbers, and $[0, e]$ is an $(I)$-semigroup.

**Theorem B.** Let $S$ be a semigroup with identity on $E_n$, $n > 1$, and $B$ a compact, connected submanifold of dimension $n-1$. If $B$ is a subsemigroup containing the identity of $S$, then:

(i) $n = 2$ or 4 and $B$ is a Lie group which is $S^1$ if $n = 2$ and $S^3$ if $n = 4$ (where $S^i$ denotes the $i$-sphere);

(ii) there exists a subsemigroup $J$ contained in the center of $S$ which is isomorphic to a semigroup of the type described in Theorem A;

(iii) the subsemigroup $J$ meets each orbit $xB = Bx$ of $B$ in exactly one point, and $J B = S$;

(iv) if 0 denotes the zero of $J$, then 0 is a zero for $S$, and $(J \setminus \{0\}) \times B$ is isomorphic to $(J \setminus \{0\}) B = S \setminus \{0\}$ in the natural way.

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1. Preliminaries. We shall use the word semigroup to mean topological semigroup, that is, a Hausdorff space with a continuous, associative multiplication. A subsemigroup will always be closed. An ideal of the semigroup $S$ is a set $A \subseteq S$ such that $SAS \subseteq A$. A zero for $S$ is an element $0$ such that $0x = x0 = 0$ for all $x \in S$.

In place of the phrase “simultaneous isomorphism and homeomorphism” we use the word iseomorphism. The symbol $\Box$ will be used to denote the null set, and $A \setminus B$ to denote the set-theoretic difference of the sets $A$ and $B$.

If $S$ is a semigroup with identity $1$, we denote by $H(1)$ the set of all elements with two-sided inverses relative to $1$. We shall refer to $H(1)$ as the maximal subgroup of the identity. In general, $H(1)$ need not be a topological group since inversion is not necessarily continuous.

Let $S$ be a semigroup, and for $x \in S$, let $T(x) = \{x, x^2, \ldots \}$. If $T(x)$ is compact, it contains exactly one idempotent $f$. If $fx = x$, then $T(x)$ is a group $[1]$.

An $(I)$-semigroup is a semigroup on a closed interval of the line such that one end point functions as a zero and the other as an identity for the semigroup. Hence, if $[0, e]$ is an $(I)$-semigroup with $0$ a zero, then $ex = xe = x$ for all $x \in [0, e]$. The complete structure of $(I)$-semigroups (and also of $(L)$-semigroups as described below) has been obtained by the authors $[2]$.

An $(L)$-semigroup is a semigroup $S$ with identity on a compact manifold with connected boundary $B$ such that $B$ is a Lie group. There is an $(I)$-semigroup $J$ contained in the center of $S$ such that $JB = S$, and such that $J$ meets each orbit $xB = Bx$ of $B$ in exactly one point. If $S$ is the two or four cell, and $B$ is $S^1$ or $S^3$ respectively, then the single exceptional orbit of $B$ is a point which acts as a zero for $S$. Obviously, this description implies $S/B$ is an $(I)$-semigroup iseomorphic under the natural projection to $J$.

2. Proof of Theorem A.

Lemma. If $S$ is a semigroup with identity such that $Q = S \setminus H(1)$ has compact closure, then $Q$ is an ideal.

Proof. Since $H(1)$ is a group, if $x \in Q$, $y \in H(1)$, then quite trivially $xy \in H(1)$. Hence, we may assume $y \in Q$. Now if $xy = g \in H(1)$, then $z = yg^{-1} \in Q$ and $xz = 1$. Multiplying successively on the left by $x$ and on the right by $z$, we have $x^n z^n = 1$. Suppose $x^n \in H(1)$ for some $n$, say $x^n = h$. Then $x(x^{n-1}h^{-1}) = (h^{-1}x^{n-1})x = 1$, so that $x \in H(1)$. Hence, $\Gamma(x)$ and $\Gamma(z)$ are compact, and we may assume, for some subsequence, $x_{\gamma(n)} \rightarrow f$, $z_{\gamma(n)} \rightarrow w$, where $f$ is idempotent. Hence, $f w = 1$. Multiplying on the left by $f$, we have $f^2 w = f$, and since $f^2 = f$, $f = 1$. This implies
Γ(x) is a group, and x∈H(1), a contradiction. Thus, \( QS \cup SQ \subseteq Q \).

We now are prepared to prove the theorem. For any \( \varepsilon > 0 \), let \( I_\varepsilon \) be a closed interval about 1 of radius \( \varepsilon \). Let \( a \) and \( b \) denote the end points of \( I_\varepsilon \). By the continuity of multiplication (which is uniform on compact sets) there exists an interval \( I_\delta \supseteq I_\varepsilon \), of radius \( \delta > 0 \), \( 1 \in I_\delta \), such that \( |ax - a|, |xa - a|, |bx - b|, |xb - b| \), are all less than \( \varepsilon \) for \( x \in I_\delta \). Hence, for \( x \in I_\delta \), \( 1 \in xI_\delta \cap I_\delta x \); that is, \( x \) has a right inverse and a left inverse in \( I_\delta \). Since the two, if they exist, must coincide, \( x \) has an inverse which we denote by \( x^{-1} \). Clearly \( x \to x^{-1} \) is continuous (i.e., \( x^{-1} \to 1 \) as \( x \to 1 \)).

Now choose a connected neighborhood \( V \) of 1 such that \( V = V^{-1} \) (since \( I_\varepsilon \) is a locally compact local group, this can always be done). Let \( G = U_x \cap V \). Then \( G \) is a one-dimensional connected Lie group, and hence is either the circle or the line. Since \( G \subseteq E \), \( G \) is the line. Clearly, \( G^- \) is either of the form \([e, f]\) or \([e, \infty]\). If it is \([e, f]\), then, since \( x^n \to e \) implies \( x^{-n} \to f \), \( ef = fe = 1 \). In this case, \([e, f]\) is a compact group, which is impossible. Hence, \( G^- = [e, \infty] \), and \( G = (e, \infty) \). Now if \( x \in G \), and \( x < 1 \), then \( x^n \to e \). It follows that \( e^2 = e \). Obviously, \( e \) is the largest idempotent different from 1, and \( ex = xe = e \) for \( x \in G \). This proves that \([e, \infty]\) is isomorphic to \([0, \infty]\) under the usual multiplication of real numbers.

Clearly \( G \) is an open subgroup of \( H(1) \) so that, trivially, \( H(1) \) is a topological group. Suppose \( x \in H(1) \setminus G \). Then \( xG \subseteq [0, e] \), so that \((xG)^-\) is a closed interval, say with end points \( g \) and \( f \). Then for some \( y \in G \), \( xy^n \to g \), \( xy^{-n} \to f \). Since \( x \in H(1) \), \( y^n \to x^{-1}g \), \( y^{-n} \to x^{-1}f \). But we have already seen that either \( y^n \to \infty \) or \( y^{-n} \to \infty \). Hence, \( H(1) = G \).

By the lemma, \([0, e] = S \setminus H(1) \) is an ideal. Moreover, since \( e \in [0, e] \), and \([0, e] \) are connected sets containing 0 and \( e \), \( e \) acts as an identity for \([0, e] \) so that \([0, e] \) is an \((I^-)\)-semigroup. This completes the proof of Theorem A.

Notice that if \( x \in [0, e] \), \( y \in [e, \infty] \), then \( xy = yx = x \) since \( e \) acts as a zero for \([e, \infty]\) and an identity for \([0, e]\).

3. **Proof of Theorem B.** Since \( B \) is a compact manifold which is a semigroup with identity, it is a Lie group \([1]\). By a theorem of Montgomery and Zippin \([3]\), \( B \) is an \((n-1)\)-sphere, and hence is \( S^1 \) or \( S^3 \). Thus, \( n = 2 \) or 4. Moreover, all orbits, except one, are homeomorphic to \( B \) (i.e., \( B \) is simply transitive on each orbit), and the exceptional orbit is a point (where we can assume \( B \) operates on the left or right of \( S \) by \( g(x) = gx \) or \( g(x) = xg \) respectively).

Let \( X \) denote the space of right orbits, \( xB \), with the usual identification topology where \( \pi: S \to X \) is the natural projection. It is also proved in \([3]\) that \( X \) has a cross section \( C \) so that \( CB = S \). It follows
that $X$ is homeomorphic to $[0, \infty)$, the half-line. We shall identify $X$ with $[0, \infty)$ where $\pi(1) = 1$ and $\pi(x_0) = 0$, where $x_0$ denotes the point in the exceptional right orbit. The proof proceeds in several short steps which we shall number for easy reference.

3.1. If $n(xB, x_0)$ denotes the index (degree) of the mapping $f_x : B \to S$ defined by $f_x(b) = xb$ relative to $x_0$ then $n(xB, x_0) = 1$ for $x \neq x_0$.

**Proof.** Let $J$ be a cross section from $x$ to 1 of $[\pi(x), \pi(1)]$, (see [3]). Then $x_0 \in JB$ so that $n(jB, x_0)$ is defined for all $j \in J$ and is constant. But $n(B, x_0) = 1$.

3.2. Define $x < y$ if $\pi(x) < \pi(y)$. Then $x < y$ if and only if $n(yB, x) = 1$.

**Proof.** Suppose $x < y$. Then there exists a cross section $J$ of $[0, \pi(x)]$ from $x_0$ to $x$, $y \in JB$. Hence, $n(yB, x) = n(yB, x_0) = 1$. On the other hand, if $n(yB, x) = 1$, then clearly $\pi(x) \neq \pi(y)$, for otherwise $x \in yB$ so that $n(yB, x)$ is not defined. Now if $x > y$, then $n(yB, x) = n(xB, x) = 0$. Hence, the only alternative is $x < y$.

3.3. If $x < y$, then $bx < by$, and $xb < yb$, $b \in B$.

**Proof.** The second statement is trivial. Now suppose $b \in B$ and $bx \in yB$. If we left multiply by $b^{-1}$, we have $x \in yB$. Hence, $bx \in yB$ for $b \in B$, and $n(byB, bx)$ is defined. Since $B$ is connected, and $gx \in gyB$ for every $g \in B$, $n(byB, bx) = n(yB, x) = 1$ which implies the result by 3.2.

3.4. For every $x \in S$, $xB = Bx$. Moreover, $X$ is a semigroup on $[0, \infty)$ with zero at 0 and identity at 1.

**Proof.** We shall show that $bx > x$ and $x > bx$ are both false. This will then tell us that $bx \in xB$. The dual argument will show $xb \in Bx$.

Assume $x > bx$. Left multiply by $b^{-1}$ to get, by 3.3,

$$b^{-1}x > x > bx.$$ 

Left multiply by $b$ successively to get

$$b^{-1}x > x > bx > b^2x > \ldots > b^n x > \ldots$$

for all positive integers $n$. Since $B$ is a compact group, $\Gamma(b)$ is a group with the property that

$$x > cx, \quad c \in \Gamma(b).$$

But $b^{-1} \in \Gamma(b)$ so that

$$b^{-1}x > x > b^{-1}x$$

which is clearly impossible.

Similarly, $x < bx$ is impossible.

Clearly, then, $X$ is a semigroup with identity $1 = \pi(B)$. Moreover, since $x_0 B = x_0$, $xx_0 B = xx_0$, so that $xx_0$ is a singular orbit. Since there
is just one, $xx_0 = x_0$ for every $x$. Since $x_0B = Bx_0$, $x_0$ is also the unique singular left orbit so that similarly $x_0x = x_0$ for $x \in S$. That is, $x_0$ is a zero for $S$. It follows that $\pi(x_0) = 0$ is a zero for $X$.

3.5. Let $D = \pi^{-1}([0, 1])$. Then $D$ and $Q = (S \setminus D)^-$ are subsemigroups.

Proof. If $x, y \in D$, then $\pi(x)\pi(y) = \pi(y)\pi(x) \in [0, 1]$. Hence, since $xy \in \pi^{-1}(\pi(xy))$, $xy \in D$. Thus, $D$ is a subsemigroup.

Now if $x, y \in Q$, a similar argument shows $xy \in Q$. (See Theorem A.)

We now complete the proof of the theorem. Let $T$ be an $(I)$-semigroup in $D$ which maps isomorphically under $\pi$ on $[0, 1]$ and such that $T$ is in the center of $D$. This can be done since obviously $D$ is an $(L)$-semigroup on the two-cell or four-cell as the case may be. Let $e$ denote the maximal idempotent of $T$ different from 1 (which exists by Theorem A). Since $T$ is isomorphic to $[0, 1] \subset [0, \infty)$, $[\pi(e), 1]$ is isomorphic to $[0, 1]$ under the usual multiplication of real numbers (Theorem A), and every element in $(\pi(e), 1]$ has an inverse in $[1, \infty)$. Let $T_0$ denote that portion of $T$ from $e$ to 1, excluding $e$ (i.e., $T_0$ is isomorphic to $(\pi(e), 1]$). For each $x \in T_0$, there exists an element $y \in Q$ such that $xy \in B$, since $xB$ has an inverse in $X$. Say $xy = b$. Let $x^{-1} = yb^{-1}$. Clearly, using a like argument on the other side, $x$ has a left and a right inverse and hence a unique inverse, and this is $x^{-1}$. Obviously $x \mapsto x^{-1}$ is one-one. Moreover, if $x_n \to 1$, then, for some subsequence, $x^{-1}_n \to b \in B$, for some $b$.

Hence $x \mapsto x^{-1}$ is continuous, and similarly, $x^{-1} \mapsto x$ is continuous (in fact, they are isomorphisms). Let $\sigma: T_0 \to S$ be the function defined by $\sigma(x) = x^{-1}$, and $T' = \sigma(T_0)$. Then, $\pi(T') = [1, \infty)$.

Since $T_0$ and $T'$ are one-parameter semigroups, for $x \in T_0, y \in T'$, either there is an element $z \in T'$ such that $y = xz^{-1}$, or there exists an element $z \in T_0$ such that $x = y^{-1}z$. Then $xy = z \in T' \cup T_0$. Moreover, $e$ being a zero for $T_0$, is clearly a zero for $T'$ also. It follows that $J = T \cup T'$ is closed under multiplication and is a cross section of $X$. Moreover, since $xb = bx$ for $x \in T, b \in B$, clearly the same is true for all $x \in J$.

Now if $x \in J, y \in S, y = x'b'$ for some $x' \in J, b' \in B$. Hence, $xy = xx'b' = x'b'x$

since $J$ is abelian and commutes elementwise with $B$. Thus, $J$ is in the center of $S$. The remainder of the theorem is immediate.

4. Other semigroups on $E_i$. If we assume that $S$ is a semigroup on $E_i = (\infty, \infty)$ with $-1, 0, 1$ playing their natural roles, then the conclusion of Theorem B follows in a similar way, where now the group $B = \{1, -1\}$. 
On the other hand, if we continue to assume $B$ connected, then $B = \{1\}$. In this case, $S$ need not be of this form, and in fact $(-\infty, 0]$ need not contain any elements of the maximal subgroup as the following example shows.

4.1. Example. For a fixed $e$, $0 < e < 1$, take $S = (-e, \infty)$ and define multiplication $x \circ y$ in $S$ as follows:

(i) for $x, y \in [0, e]$, $x \circ y = \min(x, y)$,
(ii) for $x, y \in (-e, 0]$, $x \circ y = |x| \circ |y|$, 
(iii) for $x \in [0, e]$, $y \in (-e, 0]$, $x \circ y = y \circ x = -(|x| \circ |y|)$,
(iv) for $x, y \in [e, \infty)$, $x = z_1 + e$, $y = z_2 + e$, $x \circ y = z_1 z_2 + e$,
(v) for $x \in [e, \infty)$, $y \in (-e, e]$, $x \circ y = y \circ x = y$.

Then $S$ is a semigroup on $E_1$ with a connected, compact subgroup of dimension one less, namely $\{1\}$, but no element in $(-e, 0]$ has an inverse since $x \circ y \in (-e, e]$ for $x \in (-e, 0]$, $y \in S$.

Bibliography


Tulane University