A NOTE ON TOPOLOGICAL PROPERTIES OF NORMED LINEAR SPACES[1]

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In a recent paper [2], the author established certain topological properties of infinite-dimensional normed linear spaces. A fundamental tool was the theorem of Šmulian [4] to the effect that every nonreflexive normed linear space contains a decreasing sequence of nonempty bounded closed convex sets whose intersection is empty. Thus several of the main results were obtained only for nonreflexive spaces and, using Mazur's homeomorphism [3] between L^1 and L^p, for infinite-dimensional L^p spaces. In the present note we produce an alternative tool, dealing with sequences of linearly bounded closed convex sets, whereby some of the results of [2] can be established for arbitrary infinite-dimensional normed linear spaces.

A subset of a linear space is *linearly bounded* provided it intersects each line in a bounded set. Our substitute for Šmulian's theorem is the following:

THEOREM 1. *Every infinite-dimensional normed linear space E contains a decreasing sequence C_α of unbounded but linearly bounded closed convex sets whose intersection is empty.*

PROOF. We assume without loss of generality that E is separable, and hence that E admits a sequence f_α of continuous linear functionals such that $\bigcap_1^\infty f_i^{-1}0 = \{\phi\}$. We describe first the construction of C_0.

For each i and j let $Q_i^j = f_i^{-1}[-j, j]$. Let $n_1 = 1$ and choose $p_1 \in Q_1^{n_1}$ such that $\|p_1\| = 1$. Having selected n_i and p_i for $1 \leqq i < k$, choose n_k so that $\{p_i \mid 1 \leqq i < k\} \subset Q_k^{n_k}$ and then choose $p_k \in \bigcap_1^k Q_i^{n_i}$ so that $\|p_k\| = k$. Proceeding by mathematical induction, we obtain infinite sequences n_α of positive integers and p_α of points of E. Let $C_0 = \bigcap_1^\infty Q_i^{n_i}$. Then clearly C_0 is closed and convex; and $\{p_i \mid 1 \leqq i < \infty\} \subset C_0$, so C_0 is unbounded. Now consider an arbitrary point $x \in E \sim \{\phi\}$. For some k, $f_k x = \delta \neq 0$, and then $C_0 \cap (]-\infty, \infty[x) \subset |\delta|^{-1} n_k [-1, 1]x$. Thus C_0 contains no ray through ϕ, and since C_0 is closed and convex it follows easily that C_0 is linearly bounded.

To complete the construction of C_α, we observe that since C_0 is unbounded, then by the uniform boundedness principle [1], E admits a continuous linear functional f such that $\sup_{C_0} f = \infty$. Thus with

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$C_i = C_0 \cap f^{-1}[i, \infty]$ for each i, C_α is the desired sequence of sets. Theorem 1 has been proved.

Now using Theorem 1 in lieu of Šmulian's theorem to obtain a result similar to (I 4.1) of [2], and following the arguments of §II 1 of [2], it is possible to prove the following:

THEOREM 2. *Suppose E is an infinite-dimensional normed linear space, Y is a compact subset of E, U is the unit cell $\{x \mid \|x\| \leqq 1\}$ of E, S is the unit sphere $\{x \mid \|x\| = 1\}$ of E, H is a hyperplane in E, and Q is a closed half-space bounded by H. Then*

(i) *$E \sim Y$ is homeomorphic with E.*

(ii) *There is a homeomorphism of $E \sim \text{Int } U$ onto U which is the identity on S.*

(iii) *There is a homeomorphism of period two without fixed points of E onto E which takes U onto U.*

(iv) *There is a homeomorphism of Q onto U which takes H onto S.*

It is not clear that Theorem 1 will help to extend the Isotopy Theorem of [2] to more general spaces, for considerations of boundedness enter at several points in the proof of the latter.

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