FIXED POINTS IN PRODUCTS OF ORDERED SPACES

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A topological space is said to have the fixed point property if every map (i.e. continuous function) from the space to itself leaves at least one point fixed. There is a rather well known conjecture (see e.g. [**3**]) asking if the fact that X and Y have the fixed point property implies that the product space $X \times Y$ has the property. This note contains an affirmative answer in the case that X and Y are compact ordered sets (with the order topology, of course). It is perhaps worth noting that there is an analogous result due to Ginsburg [**1**] for similarity transformations on ordered sets with the product ordered lexicographically. Since a compact ordered space has the fixed point property if and only if it is connected, we may rephrase our result to read as follows. If X and Y are compact connected ordered spaces and $Z = X \times Y$, then Z has the fixed point property. We first establish several auxiliary properties of the space Z.

LEMMA 1. *Z is unicoherent (i.e. if Z is the union of two closed and connected sets, then the intersection of the two sets is connected).*

PROOF. Suppose $Z = A \cup B$ with A and B closed and connected, but $A \cap B$ is not connected. Then $A \cap B = C \cup D$ with C and D non-null, closed, and disjoint. Since Z is normal, we have disjoint open sets P and Q with $C \subset P$ and $D \subset Q$. For each (x, y) in Z we can find an open "rectangular" neighborhood $N(x, y)$ such that

if (x, y) is in $Z - A$, $N(x, y)$ is in $Z - A$,
if (x, y) is in $Z - B$, $N(x, y)$ is in $Z - B$,
if (x, y) is in C, $N(x, y)$ is in P, and
if (x, y) is in D, $N(x, y)$ is in Q.

Since Z is compact, a finite set $\mathfrak{N}$ of these neighborhoods cover Z. We can easily find finite sets $\{x_0, x_1, \cdots, x_n\}$ and $\{y_0, y_1, \cdots, y_m\}$ in X and Y respectively such that

$$x_0 = \inf \{x \mid x \in X\} < x_1 < x_2 < \cdots < x_n = \sup \{x \mid x \in X\},$$
$$y_0 = \inf \{y \mid y \in Y\} < y_1 < y_2 < \cdots < y_m = \sup \{y \mid y \in Y\},$$

and for each i and j, R_{ij} is contained in some $N \in \mathfrak{N}$, where $R_{ij} = \{x \mid x_i \leqq x \leqq x_{i+1}\} \times \{y \mid y_j \leqq y \leqq y_{j+1}\}$.

Presented to the Society, November 18, 1955; received by the editors January 21, 1955 and, in revised form, July 20, 1955.

Now let $E=\bigcup\{R_{ij}|R_{ij}\cap A\neq\square$ (the null set)$\}$ and $F=\bigcup\{R_{ij}|R_{ij}\cap B\neq\square\}$. Clearly $(E\cap F)\subset(P\cup Q)$ and $Z=E\cup F$. By means of Urysohn's Lemma (see e.g. 1.11, p. 71 [**4**]) we can construct a map f on X to I such that $f[x_i, x_{i+1}]=[i/n, (i+1)/n]$ and a map g on Y to I with $g[y_j, y_{j+1}]=[j/m, (j+1)/m]$ (where $[a, b]$ stands for the closed interval from a to b and $I=[0, 1]$). Define h: $Z\to I^2$ by $h(x, y)=(f(x), g(y))$.

The following properties can be verified for h.

(a) $h(E)\cup h(F)=h(E\cup F)=h(Z)=I^2$.

(b) A and B connected imply that E and F are connected, and these in turn imply that $h(E)$ and $h(F)$ are connected.

(c) $h(E)$ and $h(F)$ are closed.

(d) $h(E)\cap h(F)=h(E\cap F)=h([E\cap F\cap P]\cup[E\cap F\cap Q])=h(E\cap F\cap P)\cup h(E\cap F\cap Q)$, the union of two non-null closed disjoint sets. Note that statements (a)–(d) are a contradiction to the well known fact that I^2 is unicoherent (see e.g. Corollary, p. 118 [**2**]); so that the lemma is proved.

COROLLARY. *If C is a connected subset of Z, and if $Z-C$ is connected, then $F(C)$, (the boundary of C), is also connected.*

To see that the corollary is true merely take $\overline{C}$ and $\overline{Z-C}$ as A and B in the lemma.

LEMMA 2. *If A is a closed subset of Z, then either* (a) *there is a component B of $Z-A$ such that $\pi_Y(B)=Y$ or* (b) *there is a component K of A such that $\pi_X(K)=X$ (where π_X and π_Y are the projection maps on X and Y).*

PROOF. It is easily seen that Z is locally connected; hence, components of open sets are open, and $\mathfrak{B}=\{B|B$ is a component of $Z-A\}$ is a collection of open sets. We prove the lemma by showing that if (a) is not true, then (b) must hold. Suppose therefore that $\pi_Y(B)\neq Y$ for all B in $\mathfrak{B}$. We divide the proof into two parts.

CASE I. There is a B with the property that $\pi_X(B)=X$. Since $\pi_Y(B)\neq Y$, there is a $y\in Y$ such that $(X\times\{y\})\subset(Z-B)$. Since $X\times\{y\}$ is connected, it is contained in some C, a component of $Z-B$. Let $D=Z-C$. Clearly D is connected so that by the corollary we have that $F(C)$ is connected. Since $\pi_X(B)=\pi_X(C)=X$, it is easily seen that $\pi_XF(C)=X$. As another consequence of local connectedness we have $F(C)\subset F(B)$ which is a subset of A. Since $F(C)$ is connected it is contained in a component K of A and $\pi_X(K)=X$ as was to be shown.

CASE II. For all $B\in\mathfrak{B}$ we have $\pi_XB\neq X$. For each B let

$\tilde{B}=B\cup\bigcup\{C\mid C$ is a component of $Z-B$ and $\pi_X C\neq X\}$. If $y\notin\pi_Y B$, then $X\times\{y\}\subset Z-B$ and C the component which contains $X\times\{y\}$ must satisfy $\pi_X C=X$; hence $C\not\subset\tilde{B}$ and $y\notin\pi_Y\tilde{B}$. Thus we have shown

(i) $\pi_Y\tilde{B}=\pi_Y B$.

Since B is not separated from any components of its complement, we have

(ii) $\tilde{B}$ is connected.

Since $\pi_X B\neq X$, there can be only one component C of the complement of B with $\pi_X C=X$; hence $Z-\tilde{B}$ consists of a single component of $Z-B$ and as a consequence

(iii) $Z-\tilde{B}$ is connected, and

(iv) $\tilde{B}$ is open.

Now suppose $\tilde{B}_1\cap\tilde{B}_2\neq\Box$. Then, using the notation C_i is a component of $Z-B_i$, since $B_1\cap B_2=\Box$, we have B_1 meets some C_2, B_2 meets some C_1, or some C_1 meets some C_2.

(α) If (say) $B_1\cap C_2\neq\Box$, then since B_1 is connected and $B_1\subset Z-B_2$, we have $B_1\subset C_2$. If, then, $C_1\cap B_2\neq\Box$, as above we show $B_2\subset C_1$ and using (i) we can write

$$\pi_Y B_1\subset\pi_Y(C_2)\subset\pi_Y(B_2)\subset\pi_Y(C_1)\subset\pi_Y(B_1).$$

But since π_Y is both an open and closed map, it is easily seen that all of the inclusions are proper. This contradiction shows that $C_1\cap B_2=\Box$, so that since $C_1\cup B_1$ is connected, we have $(C_1\cup B_1)\subset C_2$, and hence $\tilde{B}_1\subset C_2\subset\tilde{B}_2$.

(β) If $C_1\cap C_2\neq\Box$, and $B_1\cap C_2=\Box$, then $B_1\cap(C_2\cup B_2)=\Box$ and since $C_2\cup B_2$ is connected $(C_2\cup B_2)\subset C_1$ and (α) applies; so that we have verified

(v) $\tilde{B}_1\cap\tilde{B}_2\neq\Box$ implies $\tilde{B}_1\subset\tilde{B}_2$ or $\tilde{B}_2\subset\tilde{B}_1$.

Now if B_λ is any member of $\mathcal{B}$, we define

$$B_\lambda^*=\bigcup\{\tilde{B}\mid\tilde{B}\supset B_\lambda\}.$$

We have immediately that each B^* is open and connected. In addition each $Z-B^*$ is connected since it is the intersection of a nest of closed connected sets so that using the corollary gives

(vi) Each $F(B^*)$ is connected.

Let $\tilde{A}=Z-\bigcup\{\tilde{B}\mid B\in\mathcal{B}\}=Z-\bigcup\{B^*\mid B\in\mathcal{B}\}$. For each $x\in X$, $\{x\}\times Y$ is a compact connected set, and by use of (i), (iv), and (v) we can show $\{x\}\times Y\not\subset\bigcup\{\tilde{B}\mid B\in\mathcal{B}\}$; i.e.

(vii) $\pi_X\tilde{A}=X$.

If $\tilde{A}=M\cup N$, with M and N closed and disjoint, let $M'=M\cup\bigcup\{B^*\mid F(B^*)\cap M\neq\Box\}$ and $N'=N\cup\bigcup\{B^*\mid F(B^*)\cap N\neq\Box\}$. It is easy to see that $M'\cup N'=Z$ and (vi) insures that $M'\cap N'$ is null.

To show M' is open, given any z in M' we must exhibit an open set in M' containing z. If z is in one of the B^*'s, that B^* will do. If $z \in M$, there is an open connected set P containing z but missing N. If for some B^*, $P \cap B^* \neq \square$, then since $P \subset B^*$ implies $z \in B^*$ which is impossible, we have $P \cap F(B^*)$ is not null. This means, since $P \subset (Z-N)$ and (vi) holds, that $B^* \subset M'$ and hence $P \subset M'$, completing the proof. A similar argument shows N' is open; hence, since Z is connected, M or N is null and

(viii) $\tilde{A}$ is connected.

Since $\tilde{A} \subset A$, (vii) and (viii) show there is a component K of A such that $\pi_X(K) = X$.

THEOREM. *Z has the fixed point property.*

PROOF. Let f be any map on Z to Z. Let $A = \{(x, y) \mid \pi_Y f(x, y) = y\}$. Clearly A is closed, hence, by Lemma 2 we have either B, a component of $Z - A$, with $\pi_Y(B) = Y$, or K, a component of A, with $\pi_X(K) = X$. The first of these possibilities is untenable, however, since $B \cap \{(x, y) \mid \pi_Y f(x, y) > y\}$ and $B \cap \{(x, y) \mid \pi_Y f(x, y) < y\}$ are two non-null disjoint open sets whose union is B, a connected set. Hence K exists and since K is connected, either one of the open (in K) and disjoint sets $K \cap \{(x, y) \mid \pi_X f(x, y) > x\}$ and $K \cap \{(x, y) \mid \pi_X f(x, y) < x\}$ is null, or their union cannot be all of K. In either case there is at least one point (x', y') in K such that $\pi_X f(x', y') = x'$. But since K is a subset of A, we have $\pi_Y f(x', y') = y'$ and (x', y') is a fixed point.

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