A SPACE OF SUBSETS HAVING THE FIXED POINT PROPERTY

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In 1939 Wojdyslawski [6] asked whether the property of being a CAR* (=retract of a Tychonoff cube) is preserved from a space $X$ to the space $S(X)$ of non-null closed subsets of $X$. This has been answered affirmatively for a special case only, namely, when $X$ is a Peano space. Whether the fixed point property is preserved from $X$ to $S(X)$ is also unknown. Both properties fail to be preserved in the opposite direction [4, Corollary 1 and Corollary 10].

The object of this paper is to prove that if $X$ is a CAR* then $S(X)$ has the fixed point property. First it will be shown that if $T$ is a Tychonoff cube then $S(T)$ has the fixed point property. If the space $X$ is a CAR* it is a retract of some cube $T$ and hence [4] $S(X)$ is a retract of $S(T)$. Thus the fixed point property in $S(X)$ follows from that in $S(T)$.

The following notation will be used. If $Y$ is a space then $S(Y)$ denotes the set of non-null closed subsets of $Y$ with the usual topology [5, p. 281]. For each element $a$ in a set $A$ let $I_a$ be the unit interval; then the cartesian product

$$T = \prod \{I_a: a \in A\}$$

is a Tychonoff cube. Let $\mathfrak{A} = \{B: B$ is a finite, non-null subset of $A\}$; then for each $B$ in $\mathfrak{A}$ let

$$T_B = \prod \{I_a: a \in B\}.$$  

The projection functions are given by $\pi_a: T \to I_a$ and $\pi_B: T \to T_B$.

The proof of the theorem depends upon a collection of subsets of $S(T)$ defined, for $S$ in $\mathfrak{A}$, by $S(T, B) = S(T_B) \times P\{S(I_a): a \in B\}$. The function $r_B: S(T) \to S(T, B)$, defined by $r_B(E) = \pi_B(E) \times P\{\pi_a(E): a \in B\}$, is a retraction of $S(T)$ onto $S(T, B)$ [1, p. 68].

The fact that the system $(\mathfrak{B}, \subset)$ is a directed set is used in the following lemma. For details on nets see [2].

**Lemma.** If $\mathfrak{C}$ is a cofinal subset of $\mathfrak{A}$ and $\{E_C\}$ is a net in $S(T)$ on $\mathfrak{C}$ then $\lim_{\mathfrak{C}} \{E_C\} = E$ implies $\lim_{\mathfrak{C}} \{r_C(E_C)\} = E$.

**Proof.** Let $N(W; V)$ be a subbasic element of the topology of $S(T)$ such that $E \subseteq N(W; V)$. It is sufficient to show that the net $\{r_C(E_C)\}$ is eventually in $N(W; V)$. Now $E$ being in $N(W; V)$ means that $E \subseteq W$ and $E \cap V \neq 0$ where $W$ and $V$ are open subsets of $T$.

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Since $E$ is compact we may assume $W = \bigcup \{ W_i : i = 1, \ldots, n \}$, a finite union of basis elements of the form $W_i = \bigcup \{ U_a : a \in B_i \} \times \bigcup \{ I_a : a \in B_i \}$ where $U_a$ is an open subset of $I_a$. Let $B_0 = \bigcup \{ B_i : i = 1, \ldots, n \}$.

If $\lim E_i = E$ then there is $C_1$ in $\mathcal{C}$ such that $E_C \subseteq N(W; V)$ whenever $C$ contains $C_1$. Let $C_2$ be in $\mathcal{C}$ such that $C_1 \subseteq C_2$ and $B_0 \subseteq C_2$.

Let $C$ be any element of $\mathcal{C}$ which contains $C_2$. If $x \in r_C(E_C)$ then there is $y \in E_C$ such that $\pi_C(x) = \pi_C(y)$. Now $y \in E_C \subseteq W$ and therefore for some $i$, $y \in W_i$. Since $B_0 \subseteq C_2$, it follows that $x \in W_i$ and hence $r_C(E_C) \subseteq W$. Note that $E_C \subseteq r_C(E_C)$ so that $E_C \cap V \neq \emptyset$ implies $r_C(E_C) \cap V \neq \emptyset$. This means that $r_C(E_C) \subseteq N(W; V)$ whenever $C$ contains $C_2$, which completes the proof.

**Theorem.** The space $S(T)$ has the fixed point property.

**Proof.** It is known [3, p. 117] that $S(T)$ can be imbedded in some Tychonoff cube $R$ and we identify $S(T)$ with its homeomorphic image in $R$. Since the spaces $T_B$ and $I_a$ are all Peano spaces, it follows [4, Theorem 8] that $S(T_B)$ and $S(I_a)$ are CAR*’s, and hence each $S(T, B)$ is a CAR*. Each retraction $r_B : S(T) \to S(T, B)$ can therefore be extended to a retraction $p_B : R \to S(T, B)$.

Now let $f : S(T) \to S(T)$ be a continuous function. The composition $fp_B : R \to S(T)$ is a continuous function on a cube into itself and has a fixed point. Let $E_B$ be a fixed point of $fp_B$; then $E_B \subseteq S(T)$. This determines a net $\{ E_B \}$ in the compact space $S(T)$ on $\mathcal{C}$ and hence [2, Theorem 24] there exists a subnet $\{ E_C \}$ on a cofinal subset $\mathcal{C}$ of $\mathcal{C}$ with a limit $E$ in $S(T)$. By the lemma $\lim E = E$ implies $\lim E_C = E$. Since $f$ is continuous this means that $\lim f(E_C) = f(E)$ but $f(E_C) = fpc(E_C) = E_C$. Hence $f(E) = E$ and the theorem is proved.

**Corollary.** If $X$ is a CAR* then $S(X)$ has the fixed point property.

**References**


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