# A SPACE OF SUBSETS HAVING THE FIXED POINT PROPERTY[1]

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In 1939 Wojdyslawski [6] asked whether the property of being a CAR* (=retract of a Tychonoff cube) is preserved from a space $X$ to the space $S(X)$ of non-null closed subsets of $X$. This has been answered affirmatively for a special case only, namely, when $X$ is a Peano space. Whether the fixed point property is preserved from $X$ to $S(X)$ is also unknown. Both properties fail to be preserved in the opposite direction [4, Corollary 1 and Corollary 10].

The object of this paper is to prove that if $X$ is a CAR* then $S(X)$ has the fixed point property. First it will be shown that if $T$ is a Tychonoff cube then $S(T)$ has the fixed point property. If the space $X$ is a CAR* it is a retract of some cube $T$ and hence [4] $S(X)$ is a retract of $S(T)$. Thus the fixed point property in $S(X)$ follows from that in $S(T)$.

The following notation will be used. If $Y$ is a space then $S(Y)$ denotes the set of non-null closed subsets of $Y$ with the usual topology [5, p. 281]. For each element $a$ in a set $A$ let $I_a$ be the unit interval; then the cartesian product $T=P\{I_a: a\in A\}$ is a Tychonoff cube. Let $\mathcal{B}=\{B: B$ is a finite, non-null subset of $A\}$; then for each $B$ in $\mathcal{B}$ let $T_B=P\{I_a: a\in B\}$. The projection functions are given by $\pi_a: T\to I_a$ and $\pi_B: T\to T_B$.

The proof of the theorem depends upon a collection of subsets of $S(T)$ defined, for $B$ in $\mathcal{B}$, by $S(T, B)=S(T_B)\times P\{S(I_a): a\notin B\}$. The function $r_B$: $S(T)\to S(T, B)$, defined by $r_B(E)=\pi_B(E)\times P\{\pi_a(E): a\notin B\}$, is a retraction of $S(T)$ onto $S(T, B)$ [1, p. 68].

The fact that the system $(\mathcal{B}, \subset)$ is a directed set is used in the following lemma. For details on nets see [2].

LEMMA. *If $\mathcal{C}$ is a cofinal subset of $\mathcal{B}$ and $\{E_C\}$ is a net in $S(T)$ on $\mathcal{C}$ then* $\lim_{\mathcal{C}} \{E_C\}=E$ *implies* $\lim_{\mathcal{C}} \{r_C(E_C)\}=E$.

PROOF. Let $N(W; V)$ be a subbasic element of the topology of $S(T)$ such that $E\in N(W; V)$. It is sufficient to show that the net $\{r_C(E_C)\}$ is eventually in $N(W; V)$. Now $E$ being in $N(W; V)$ means that $E\subset W$ and $E\cap V\neq 0$ where $W$ and $V$ are open subsets of $T$.

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Since $E$ is compact we may assume $W=\bigcup\{W_i: i=1, \cdots, n\}$, a finite union of basis elements of the form $W_i=P\{U_a: a\in B_i\} \times P\{I_a: a\notin B_i\}$ where $U_a$ is an open subset of $I_a$. Let $B_0=\bigcup\{B_i: i=1, \cdots, n\}$.

If $\lim_{\mathfrak{C}} \{E_C\}=E$ then there is $C_1$ in $\mathfrak{C}$ such that $E_C\in N(W; V)$ whenever $C$ contains $C_1$. Let $C_2$ be in $\mathfrak{C}$ such that $C_1\subset C_2$ and $B_0\subset C_2$.

Let $C$ be any element of $\mathfrak{C}$ which contains $C_2$. If $x\in r_C(E_C)$ then there is $y\in E_C$ such that $\pi_C(x)=\pi_C(y)$. Now $y\in E_C\subset W$ and therefore for some $i$, $y\in W_i$. Since $B_0\subset C_2$, it follows that $x\in W_i$ and hence $r_C(E_C)\subset W$. Note that $E_C\subset r_C(E_C)$ so that $E_C\cap V\neq 0$ implies $r_C(E_C) \cap V\neq 0$. This means that $r_C(E_C)\in N(W; V)$ whenever $C$ contains $C_2$, which completes the proof.

THEOREM. *The space $S(T)$ has the fixed point property.*

PROOF. It is known [**3**, p. 117] that $S(T)$ can be imbedded in some Tychonoff cube $R$ and we identify $S(T)$ with its homeomorphic image in $R$. Since the spaces $T_B$ and $I_a$ are all Peano spaces, it follows [**4**, Theorem 8] that $S(T_B)$ and $S(I_a)$ are CAR*'s, and hence each $S(T, B)$ is a CAR*. Each retraction $r_B$: $S(T)\to S(T, B)$ can therefore be extended to a retraction $p_B$: $R\to S(T, B)$.

Now let $f$: $S(T)\to S(T)$ be a continuous function. The composition $fp_B$: $R\to S(T)\subset R$ is a continuous function on a cube into itself and has a fixed point. Let $E_B$ be a fixed point of $fp_B$; then $E_B\in S(T)$. This determines a net $\{E_B\}$ in the compact space $S(T)$ on $\mathfrak{B}$ and hence [**2**, Theorem 24] there exists a subnet $\{E_C\}$ on a cofinal subset $\mathfrak{C}$ of $\mathfrak{B}$ with a limit $E$ in $S(T)$. By the lemma $\lim_{\mathfrak{C}} \{E_C\}=E$ implies $\lim_{\mathfrak{C}} \{r_C(E_C)\}=E$. Since $f$ is continuous this means that $\lim_{\mathfrak{C}} \{fr_C(E_C)\}=f(E)$ but $fr_C(E_C)=fp_C(E_C)=E_C$. Hence $f(E)=E$ and the theorem is proved.

COROLLARY. *If $X$ is a CAR* then $S(X)$ has the fixed point property.*

## REFERENCES


1. C. E. Capel, *Space of subsets and cartesian products*, Bull. Amer. Math. Soc. Abstract 61-1-117.
2. J. L. Kelley, *Convergence in topology*, Duke Math. J. vol. 17 (1950) pp. 277–283.
3. ———, *General topology*, New York, Van Nostrand, 1955.
4. W. L. Strother, *Fixed points, fixed sets, and M-retracts*, Duke Math. J. vol. 22 (1955) pp. 551–556.
5. ———, *Multi-homotopy*, Duke Math. J. vol. 22 (1955) pp. 281–286.
6. M. Wojdyslawski, *Retractes absolus et hyperespaces des continus*, Fund. Math. vol. 32 (1939) pp. 184–192.


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