# NOTE ON LIE ALGEBRA KERNELS IN CHARACTERISTIC $p$

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Let $K$ and $L$ be Lie algebras over a field $F$. Let $D(K)$ denote the derivation algebra of $K$, and $I(K)$ the ideal consisting of the inner derivations of $K$. If $\phi$ is a homomorphism of $L$ into $D(K)/I(K)$ then $\phi$ defines what is called the structure of an $L$-kernel on $K$. The $L$-kernel $K$ is said to be extendible if there exists a Lie algebra extension with kernel $K$ and image $L$ which induces the given $L$-kernel structure on $K$ in the natural fashion. The $L$-kernels with a fixed $L$-module $C$ as common center can be partitioned into equivalence classes, two kernels being equivalent if they differ (in the sense of a certain composition of kernels) by an extendible kernel. It has been shown in [2] that these equivalence classes of $L$-kernels constitute a vector group over $F$ which is canonically isomorphic with the 3-dimensional cohomology group $H^3(L, C)$. In particular, every $L$-kernel determines a 3-dimensional cohomology class which is called the obstruction of the kernel and whose vanishing is equivalent to the extendibility of the kernel.

A cohomology class $u$ for the finite dimensional Lie algebra $L$ in the finite dimensional $L$-module $C$ is said to be effaceable if there exists a finite dimensional $L$-module $C'$ containing $C$ and such that the canonical image of $u$ in $H(L, C')$ is 0. It is known from [2] and [3] that every effaceable 3-dimensional cohomology class for the finite dimensional Lie algebra $L$ in a finite dimensional $L$-module $C$ is the obstruction of a finite dimensional $L$-kernel $K$ with center $C$. Moreover, in the case of characteristic 0, it has been shown in [3] that, conversely, the obstruction of a finite dimensional kernel is effaceable. The proof depends heavily on the structure and representation theory of Lie algebras of characteristic 0 and therefore breaks down completely in characteristic $p \neq 0$. The purpose of this note is to prove this result in the case of characteristic $p \neq 0$. When this is combined with the known results we have just mentioned there results the following theorem for arbitrary characteristic.

THEOREM. *Let $L$ be a finite dimensional Lie algebra and let $C$ be a finite dimensional $L$-module. Then an element of $H^3(L, C)$ is the obstruction of a finite dimensional $L$-kernel if and only if it is effaceable.*

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From now on all the Lie algebras and modules we shall consider will be assumed to be defined over a field $F$ of characteristic $p\neq 0$. Our principal tool is the universal enveloping algebra of a Lie algebra. The essential feature of our proof is that we augment a given $L$-kernel $K$ to a *restricted* $L^*$-kernel $K^*$, where $L^*$ and $K^*$ are finite dimensional restricted Lie algebras containing $L$ and $K$, respectively. The effaceability of the obstruction of $K$ will then follow from the fact that the obstruction of the $L^*$-kernel $K^*$ is effaceable. The construction of the restricted Lie algebras $K^*$ and $L^*$ will proceed within the universal enveloping algebras of $K$ and $L$, respectively, and will depend on the consideration of the $p$th power map. Essentially, the technique we use in our construction is the same as that of Iwasawa [**4**] and Jacobson [**5**], although we are applying it here to a considerably more complicated situation.

For later reference, we collect a few well known facts concerning derivations of algebras of characteristic $p$.

(1) If $t$ is a derivation of a Lie algebra or of an associative algebra of characteristic $p$, then $t^p$ is also a derivation.

Let $U$ be either a Lie algebra or an associative algebra, and let $u\in U$. The inner derivation of $U$ which is effected by the element $u$ will always be denoted by $D_u$. If $U$ is an associative algebra we have $D_u(x)=ux-xu$. If $U$ is a Lie algebra we have $D_u(x)=[u, x]$.

(2) If $U$ is an associative algebra of characteristic $p$ and if $u\in U$ then the $p$th power of the inner derivation effected by $u$ coincides with the inner derivation effected by $u^p$.

(3) Let $u$ and $v$ be elements of an associative algebra of characteristic $p$. Then $\sum_{i=0}^{p-1} u^i v u^{p-1-i}=D_u^{p-1}(v)$.

Now let $K$ be a Lie algebra over the field $F$ of characteristic $p$, and let $R_K$ denote the universal enveloping algebra of $K$. We define $P_K$ as the subspace of $R_K$ which is spanned by all elements of the form $x^{p^n}$, where $x\in K$ and $n=0, 1, \cdots$ (for $n=0$, $x^{p^n}$ is to be interpreted as $x$).

LEMMA 1. *Let $t$ be a derivation of $R_K$ which maps $K$ into itself. Then $t(P_K)\subset K$.*

PROOF. We have to show that, for every $x\in K$ and every nonnegative integer $n$, we have $t(x^{p^n})\in K$. For $n=0$, this holds by assumption. Generally, write $y$ for $x^{p^n}$. Using (3), we have $t(x^{p^{n+1}}) =t(y^p)=\sum_{i=0}^{p-1} y^i t(y)y^{p-1-i}=D_y^{p-1}(t(y))$. By (2), we have $D_y=D_x^{p^n}$, so that $t(x^{p^{n+1}})=D_x^{p^n(p-1)}(t(y))$. The restriction of $D_x$ to $K$ is the inner derivation of the Lie algebra $K$ which is effected by the element $x\in K$. In particular, $D_x(K)\subset K$, whence our last result shows, by induction on $n$, that $t(x^{p^n})\in K$, for all $x\in K$ and all $n$. This proves Lemma 1.

LEMMA 2. *$P_K$ is closed under the commutation $[u, v]=uv-vu$ and the map $u\to u^p$, and thus is a restricted Lie algebra with these operations. Furthermore, if $(x_i)$ is a basis for $K$ over $F$, the elements $x_i^{p^n}$ $(n=0, 1, \cdots)$ constitute a basis for $P_K$ over $F$.*

PROOF. Let $y=x^{p^n}$, with $x\in K$. By (2), $D_y=D_x^{p^n}$ and therefore maps $K$ into itself. By Lemma 1, we conclude that $D_y(P_K)\subset K$. Hence $P_K$ is closed under the commutation, and moreover we have $[P_K, P_K]\subset K$. Next we show that if $u\in P_K$ then also $u^p\in P_K$. This is evident from the definition of $P_K$ whenever $u$ is in $Fx^{p^n}$, with some $x\in K$. Hence it suffices to prove that if $u$, $u^p$, $v$, $v^p$ are all in $P_K$ then also $(u+v)^p\in P_K$. For this purpose we recall that if $u$ and $v$ are elements of an associative algebra of characteristic $p$ one has $(u+v)^p =u^p+v^p+s(u, v)$, where $s$, as a function of two independent noncommutative variables, is a certain sum of multiple commutators in these variables (a short proof of this result of Jacobson's will be found on p. 560 of [1]). In our case, since $u$ and $v$ are in $P_K$ and $[P_K, P_K]\subset K$, it follows therefore that $s(u, v)\in K$, whence $(u+v)^p\in P_K$. Finally, observe that this same argument shows that the subspace of $P_K$ which is spanned by the elements of the form $x_i^{p^n}$ is closed under the map $u\to u^p$, whence we conclude that it coincides with $P_K$. Since the elements $x_i^{p^n}$ are linearly independent, this proves the last assertion of Lemma 2.

Now let $L$ be a finite dimensional Lie algebra over $F$, and assume that $K$ is a finite dimensional $L$-kernel. Let $x\to t_x$ be a linear map of $L$ into the derivation algebra $D(K)$ of $K$ which induces the given kernel structure. This means that, for every $x\in L$, the derivation $t_x$ of $K$ belongs to the coset $\phi(x)$, where $\phi$ is the given homomorphism of $L$ into $D(K)/I(K)$. We can therefore find an alternating bilinear map $\tau$ of $(L, L)$ into $K$ such that, for all $x$, $x'$ in $L$, $[t_x, t_{x'}]=t_{[x,x']}+D_{\tau(x,x')}$.

We recall that every derivation of $K$ can be extended in one and only one way to a derivation of the associative algebra $R_K$. Moreover, an inner derivation $D_x$ of $K$ extends to the inner derivation of $R_K$ which is effected by $x$. It follows that no confusion can arise if we identify every derivation of $K$ with its canonical extension to a derivation of $R_K$. In order to keep our notation within reasonable bounds, we shall make this identification whenever it is convenient. In particular, it follows from the uniqueness of the extensions of derivations from $K$ to $R_K$ that the relation $[t_x, t_{x'}]=t_{[x,x']}+D_{\tau(x,x')}$ still holds when these derivations are understood to be the extensions to $R_K$ of the original derivations of $K$. Moreover, when thus extended, each $t_x$ induces on $P_K$ a *restricted* derivation, with respect to the structure of $P_K$ as a restricted Lie algebra. Indeed, a derivation $t$ of a restricted Lie algebra $M$ with $p$-map $m \to m^{[p]}$ is said to be a restricted derivation if, for every $m \in M$, $t(m^{[p]}) = D_m^{p-1}(t(m))$; and we have already seen from (3) (at the beginning of our proof of Lemma 1) that if $t$ is any derivation of $R_K$ and $u \in R_K$ then $t(u^p) = D_u^{p-1}(t(u))$.

If $u \in R_K$, we shall call a polynomial of the form $u^{p^k} + \gamma_1 u^{p^{k-1}} + \cdots + \gamma_k u$, where the $\gamma_i \in F$, and $k > 0$, a $p$-polynomial in $u$. If $u \in K$ it follows from (2) that the inner derivation effected by any $p$-polynomial in $u$ maps $K$ into itself. It follows that, with $x_1, \cdots, x_m$ a basis for $K$ over $F$, we can find a $p$-polynomial $u_i$ in $x_i$ which commutes with every element of $K$ and therefore belongs to the center of $R_K$. Evidently, $u_i \in P_K$. Now let $z_1, \cdots, z_n$ be a basis for $L$ over $F$. Since, by Lemma 1, $t_{z_1}$ maps $P_K$ into $K$ which is finite dimensional, there exists a $p$-polynomial $u_{i,1}$ in $u_i$ which is annihilated by $t_{z_1}$. Clearly, $u_{i,1}$ is also a $p$-polynomial in $x_i$. Next, there is a $p$-polynomial $u_{i,2}$ in $u_{i,1}$ which is annihilated by $t_{z_2}$, and also by $t_{z_1}$, of course. Continuing this construction, we finally obtain a $p$-polynomial $q_i = u_{i,n}$ in $x_i$ which is annihilated by each $t_{z_j}$, and hence also by each $t_z$ with $z \in L$. Let $Q_K$ denote the subspace of $P_K$ which is spanned by the elements $q_i^{p^r}$, $i = 1, \cdots, m$; $r = 0, 1, \cdots$. Evidently, $Q_K$ is a restricted ideal of $P_K$ and is contained in the center of $P_K$ (even of $R_K$). We put $K^* = P_K/Q_K$. Since our $p$-polynomials are all of degree greater than 1, we have $Q_K \cap K = (0)$, so that we may identify $K$ with a Lie subalgebra of the restricted Lie algebra $K^*$. Furthermore, it is clear from our construction and the last assertion of Lemma 2 that $K^*$ is finite dimensional. It is also evident from our construction that the map $x \to t_x$ induces a linear map $x \to t_x^*$ of $L$ into the restricted derivation algebra of $K^*$ such that, for all $y, z$ in $L$, $[t_y^*, t_z^*] - t_{[y,z]}^*$ is the inner derivation of $K^*$ which is effected by the element $\tau(y, z) \in K$.

Now construct the restricted Lie algebra $P_L$. For our basis $z_1, \cdots, z_n$ of $L$, define $t^*(z_j^{p^r}) = (t_{z_j}^*)^{p^r}$, and extend this linearly to obtain a linear map $t^*$ of $P_L$ into the restricted derivation algebra of $K^*$. (In order to see that the $p$th power of a restricted derivation is still a restricted derivation, note that, in the universal enveloping algebra, the property for a derivation $t$ to be restricted can be expressed by: $t(x^{[p]}) = t(x^p)$, where $x^{[p]}$ is the $p$-image of $x$ in the restricted Lie algebra and $x^p$ is the $p$th power of $x$ in the universal enveloping algebra). We claim that, for any $a$ and $b$ in $P_L$, $[t^*(a), t^*(b)] - t^*([a, b])$ is an inner derivation of $K^*$, effected by an element of $K$. It is clearly sufficient to prove this in the case where $a$ and $b$ are of the form $z_j^{p^r}$. Our claim holds in virtue of what we have seen above when $r=0$, both in $a$ and in $b$. Making an induction on the degree, we may deduce the result to be proved as soon as we have shown that our claim holds for $a^p$ and $b$ whenever it holds for $a$ and $b$. Then we have $[t^*(a^p), t^*(b)] = [t^*(a)^p, t^*(b)] = D_{t^*(a)}^{p-1}([t^*(a), t^*(b)])$. If our claim holds for $a$ and $b$, this last derivation is $D_{t^*(a)}^{p-1}(t^*([a, b]) + D_x)$, where $D_x$ denotes the inner derivation of $K^*$ which is effected by some element $x \in K$. Now $D_{t^*(a)}^{p-1}(D_x) = D_y$, where $y = t^*(a)^{p-1}(x)$. By definition, $t^*(a)$ maps $K$ into itself, whence $y \in K$. We have shown that $[t^*(a^p), t^*(b)] = D_{t^*(a)}^{p-1}(t^*([a, b])) + D_y$, with some $y \in K$. If $a = z_j^{p^r}$ the first of the derivations on the right may be written in the form $D_{t^*(z_j)}^{p^r(p-1)}(t^*([a, b]))$. From the beginning of our proof of Lemma 2, we know that $[a, b] \in L$. Hence we see that our derivation differs from $t^*(D_{z_j}^{p^r(p-1)}([a, b]))$ only by an inner derivation effected by an element of $K$. Since this last derivation is equal to $t^*(D_a^{p-1}([a, b])) = t^*([a^p, b])$, we conclude that our claim holds for $a^p$ and $b$. Thus we have indeed shown that $[t^*(a), t^*(b)] - t^*([a, b])$ is an inner derivation effected by an element of $K$, for all $a$ and $b$ in $P_L$.

Next, we shall show that, for every $a \in P_L$, the derivation $t^*(a)^p - t^*(a^p)$ is also an inner derivation effected by an element of $K$. If $a \in Fz_j^{p^r}$, this derivation is evidently 0. Hence it suffices to show that if the result holds for $a$ and for $b$ it also holds for $a+b$. Now we observe that $t^*(a+b)^p - t^*((a+b)^p) = t^*(a)^p - t^*(a^p) + t^*(b)^p - t^*(b^p) + s(t^*(a), t^*(b)) - t^*(s(a, b))$. Recalling that $s$ is a certain sum of multiple commutators, we may conclude from the result concerning commutators of derivations $t^*(a)$, etc., that the difference of the two terms involving $s$ is an inner derivation effected by an element of $K$. Hence our result holds for $a+b$ whenever it holds for $a$ and for $b$. This completes the proof of the second property of $t^*$.

In particular, it follows that $t^*$ induces a restricted homomorphism of the restricted Lie algebra $P_L$ into the restricted Lie algebra $D_0(K^*)/I(K^*)$, where $D_0(K^*)$ is the restricted derivation algebra of $K^*$ and $I(K^*)$ is the restricted ideal consisting of the inner derivations of $K^*$.

As before, using that $K^*$ and $L$ are finite dimensional, we can find a $p$-polynomial $r_j$ in $z_j$ which belongs to the center of $P_L$ and is such that $t^*(r_j) = 0$. Let $Q_L$ denote the subspace of $P_L$ which is spanned by the elements $r_j^{p^r}$, $j = 1, \cdots, n$; $r = 0, 1, \cdots$. Then we see as before that $P_L/Q_L$ is a finite dimensional restricted Lie algebra $L^*$ and that $L$ may be identified with a subalgebra of $L^*$. Moreover, $t^*$ induces a linear map of $L^*$ into $D_0(K^*)$ (still denoted $t^*$) such that the derivations $[t^*(a), t^*(b)]-t^*([a, b])$ and $t^*(a)^p-t^*(a^p)$ are inner derivations of $K^*$ effected by elements of $K$, for all $a$ and $b$ in $L^*$. In particular, if $a$ and $b$ are in $L$, then $[t^*(a), t^*(b)]-t^*([a, b])$ is the inner derivation of $K^*$ which is effected by the element $\tau(a, b)$ of $K$. Clearly, $t^*$ induces on $K^*$ the structure of a restricted $L^*$-kernel, in the sense of [2].

In order to be in a position to apply the cohomology theory of restricted Lie algebra kernels, we must make a slight modification of our kernel $K^*$. Let $C^*$ denote the center of $K^*$. Select any *p-semilinear* map $\rho$ of $K^*$ into $C^{*p}$ (i.e., for $b$, $c$ in $K^*$ and $\beta$, $\gamma$ in $F$, $\rho(\beta b+\gamma c) =\beta^p\rho(b)+\gamma^p\rho(c)$) such that $\rho(c)=c^p$ whenever $c\in C^*$. Now define a new $p$-map $a\rightarrow a^{[p]}$ in $K^*$ by setting $a^{[p]}=a^p-\rho(a)$. It is well known and easy to check that this is still a legitimate $p$-map with which $K^*$ is a restricted Lie algebra. Furthermore, if a derivation of $K^*$ is a restricted derivation with respect to the original $p$-map $a\rightarrow a^p$ then it is also a restricted derivation with respect to the new $p$-map $a\rightarrow a^{[p]}$. For this new $p$-map we have $C^{*[p]}=(0)$, i.e., $C^*$ is strongly abelian. Hence $K^*$ has now been given the structure of a restricted $L^*$-kernel with strongly abelian center $C^*$, so that the cohomology theory of [2] applies to $K^*$.

We must now refer to pp. 709–713 of [2], where we have given a construction attaching to the restricted $L^*$-kernel $K^*$ with strongly abelian center $C^*$ a certain 3-dimensional *restricted* cohomology class for $L^*$ in $C^*$. Since $L^*$ and $C^*$ are finite dimensional, this restricted cohomology class is effaceable. In fact, this follows easily from the definition of the restricted cohomology group (see [1]) noting that the *restricted* universal enveloping algebra of a finite dimensional restricted Lie algebra is finite dimensional. (Actually, if $U_{L^*}$ is the restricted universal enveloping algebra of $L^*$, our cohomology class is effaced in the module of all linear maps of $U_{L^*}$ into $C^*$).

Let $h$ be the canonical image in the *ordinary* cohomology group $H^3(L^*, C^*)$ of the restricted cohomology class attached to the restricted $L^*$-kernel $K^*$. Then $h$ is evidently still effaceable. Moreover, the construction we have referred to above shows that the cohomology class $h$ is represented by an alternating cocycle $f$ for $L^*$ in $C^*$ such that, for $x$, $y$, $z$ in $L$, $f(x, y, z)=t^*(x)(\tau(y, z))-t^*(y)(\tau(x, z)) +t^*(z)(\tau(x, y))-\tau([x, y], z)+\tau([x, z], y)-\tau([y, z], x)$ (see p. 713 of [2]). But this means precisely that the restriction of $f$ to $L$ (which actually takes values in $C\subset C^*$) represents the cohomology class $k$, say, in $H^3(L, C)$, which is the obstruction of our given $L$-kernel $K$, (see p. 702 of [2]). Hence the canonical image of $k$ in $H^3(L, C^*)$ coincides with the restriction of $h$ to $L$. Since $h$ is effaceable, so is its restriction to $L$. Hence the canonical image of $k$ in $H^3(L, C^*)$ is effaceable, whence also $k$ is effaceable. This completes the proof of our theorem.

## References

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