METRIZABILITY OF DECOMPOSITION SPACES

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1. Introduction. Let \( f \) be a quasicompact continuous mapping of a metric space \( S \) onto a topological space \( E \); that is, let \( f \) be a mapping of \( S \) onto \( E \) such that a subset \( Y \) of \( E \) is open (closed) in \( E \) if and only if \( f^{-1}(Y) \) is open (closed) in \( S \). As is well known,\(^1\) \( E \) is then homeomorphic, in a natural way, to the hyperspace of the decomposition of \( S \) into the disjoint nonempty sets \( F_p = f^{-1}(p) \) (\( p \in E \)), while conversely the hyperspace of every decomposition of \( S \) into disjoint nonempty sets \( F_p \) arises in this way, essentially uniquely. The object of this paper is to determine conditions under which \( E \) will be metrizable. Results of this nature have been known for some time, particularly for the special case in which \( f \) is closed (i.e., \( f(X) \) is closed in \( E \) whenever \( X \) is closed in \( S \))—or, equivalently, the case in which the decomposition \( \{F_p\} \) is upper semi-continuous (i.e., if \( F_p \subseteq U \) where \( U \) is open, there exists an open set \( V \) such that \( F_p \subseteq V \) and every \( F_q \) meeting \( V \) is contained in \( U \)).\(^2\) But it seems to have escaped notice that the problem has a simple complete answer in this case (Theorem 1), without any assumptions of compactness or separability.\(^3\) We shall also improve some known results about quasicompact images of locally compact spaces (Theorems 2 and 3), and obtain a criterion (Theorem 4) for the metrizability of \( E \) when \( f \) is open (i.e., when the decomposition is lower semi-continuous). Here the conditions we obtain are sufficient but not necessary; however, we show by examples that they are not superfluous. Finally we apply Theorem 1 to obtain a simple description of the case in which \( f \) is both open and closed (i.e., when the decomposition is continuous); it turns out that \( E \) is then always metrizable, with a quite convenient metric.

2. Closed mappings.

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\(^1\) See [2, p. 65; 5; 10]. In the terminology of [2] “quasicompact continuous” = “stark-stetig.” In the terminology of Bourbaki, \( E \) is the space \( S/R \) where \( R \) is the equivalence relation corresponding to \( f \) (i.e., \( xRy \iff f(x) = f(y) \)).

\(^2\) See [9, p. 123; 4]. The term “upper semi-continuous decomposition” has been used in the literature with several different meanings, but that given above seems to be generally adopted now. It is equivalent to “stetig Zerlegung” in [2], except that we do not require \( F_p \) to be closed.

\(^3\) Another, independent, proof of Theorem 1 has appeared after this paper was written; see K. Morita and S. Hanai, Closed mappings and metric spaces, Proc. Japan Acad. vol. 32 (1956) pp. 10–14.
Theorem 1. Let $f$ be a closed continuous mapping of a metric space $S$ onto a topological space $E$. Then the following statements are all equivalent:

(i) $E$ satisfies the first countability axiom.
(ii) For each $p \in E$, $f^{-1}(p)$ has a compact frontier (in $S$).
(iii) $E$ is metrizable.

Remark. $E$ is in any case a $T_1$ space, for each $p \in E$ is of the form $f(x)$ where $(x)$ is closed. Hence $F_p = f^{-1}(p)$ is closed in $S$, for each $p \in E$.

It is enough to prove (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii). Let $\{ W_n(p) \}$ ($n = 1, 2, \ldots$) be a countable basis of open neighborhoods of $p \in E$. If $Fr(F_p)$ is not compact, there is a sequence $\{ x_n \}$ ($n = 1, 2, \ldots$) of points of $Fr(F_p)$ having no cluster point in $Fr(F_p)$, and consequently no cluster point in $S$. Now $f^{-1}(W_n(p))$ is an open set containing $F_p$; as $F_p$ is closed, $x_n \in f^{-1}(W_n(p))$, and so there exists $y_n \in S - F_p$ such that $y_n \in f^{-1}(W_n(p))$ and $\rho(x_n, y_n) < 1/n$, $\rho$ denoting the distance in $S$. Let $Y = \{ y_n \}$; $Y$ is closed, since the sequence $\{ y_n \}$ has no cluster point in $S$ (else the sequence $\{ x_n \}$ would). Hence $Q = f(Y)$ must be closed in $E$. By construction, $p \notin Q$; yet $p \notin \overline{Q}$ since $W_n(p)$ meets $Q$ in $f(y_n)$, and this contradicts the closedness of $Q$.

(ii) $\Rightarrow$ (iii) The case in which the sets $F_p$ are themselves compact has been established by S. Hanai, and the present proof uses the same idea as his. We construct for each $p \in E$ a sequence $\{ W_n(p) \}$ ($n = 1, 2, \ldots$) of open sets and prove (1) $W_1(p) \supset W_2(p) \supset \cdots$, (2) $\bigcap_1^n W_n(p) = (p)$, (3) the sets $W_n(p)$ form a basis at $p$, (4) given a positive integer $n$ and a point $p$ of $E$, there exists $m$ such that whenever $W_m(q)$ meets $W_n(p)$ we have $W_m(q) \subseteq W_n(p)$. By a theorem of Mrs. Frink [3], $E$ will then be metrizable.

For any set $X \subseteq S$, write $S(X, \epsilon)$ to denote the open $\epsilon$-neighborhood $\{ y \mid \rho(y, X) < \epsilon \}$ of $X$. (If $X = \emptyset$, $S(X, \epsilon) = \emptyset$.) Now write $N^*_p = S(\text{Fr}(F_p), 1/n)$, $U^*_n = N^*_n \cup \text{Int}(F_p)$, $V^*_n = \bigcup \{ F_q \mid F_q \subseteq U^*_n \}$, and $W_n(p) = f(V^*_n) = \{ q \mid F_q \subseteq U^*_n \}$. Thus $W_n(p) = E - f(S - U^*_n)$; and, as $U^*_n$ is open, it follows from the closedness of $f$ that $W_n(p)$ and so $V^*_n$ are open. Clearly $U^*_n \supset V^*_n \supset F_p$, so $W_n(p) \not\supset p$. The assertion (1) is trivial, and (2) follows from (3) since $E$ is $T_1$. To prove (3), let $G$ be any open set (in $E$) containing $p$; then $\text{Fr}(F_p) \subseteq F_p \subseteq f^{-1}(G)$, and so the closed sets $\text{Fr}(F_p)$, $S - f^{-1}(G)$, are disjoint. As $\text{Fr}(F_p)$ is com-

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4 The implication (iii) $\Rightarrow$ (ii) is due to Vaštefn [7]. If $S$ is separable, the implication (i) $\Rightarrow$ (iii) is implicit in [10].

5 See [4]; another proof is implicit in [11, p. 367]. The general case can be deduced from the $F_c$-compact case; but, as the proof in [4] has a gap and that in [11] uses different basic definitions, it seems preferable to give a direct argument.
pact, the distance \( d = \rho \{ \text{Fr} (F_p), S - f^{-1}(G) \} \) is positive. Take \( n > 1/d \);
then \( N^n_p \subseteq f^{-1}(G) \), and consequently \( f^{-1}(G) \supseteq U^n_p \supseteq V^n_p \), so that \( G \supseteq W_n(p) \), q.e.d.

To prove (4), suppose \( p \) and \( n \) given. If \( \text{Int} (F_p) \neq \emptyset \), pick a point \( x_p \in \text{Int} (F_p) \) arbitrarily. Now take \( m \) so large that

(a) \( m > 2n \),

(b) if \( \text{Fr} (F_p) \neq \emptyset \), \( \rho (\text{Fr} (F_p), S - V^n_p) > 2/m \),

(c) if \( \text{Int} (F_p) \neq \emptyset \), \( S(x_p, 1/m) \subseteq V^n_p \).

Suppose \( W_m(q) \) meets \( W_m(p) \) in \( r \), say, where we assume that \( q \neq p \).
We first show that \( F_q \subseteq V^n_p \). Take \( w \in F_q \subseteq V^n_q \supseteq U^n_q \cap U^n_p \). If \( w \in \text{Int} (F_q) \), then \( F_q \) meets \( V^n_p \) and consequently \( F_q \subseteq V^n_q \subseteq V^n_p \) as required. Since \( w \in U^n_q = N^n_q \cup \text{Int} (F_q) \), we may assume now that \( w \in N^n_q \). Again, if \( w \in \text{Int} (F_p) \), then \( x_p \) exists; also \( F_p \) meets \( V^n_q \) and so \( F_p \subseteq V^n_q \); hence \( x_p \in V^n_q \subseteq \text{Int} (F_q) \cup N^n_q \), and as \( x_p \in F_q \) (for \( p \neq q \)) we have \( x_p \in N^n_q \), which violates condition (c). Hence \( w \in N^n_q \cap N^n_p \), and there exist points \( x \in \text{Fr} (F_q) \), \( y \in \text{Fr} (F_q) \), such that \( \rho (w, x) < 1/m \) and \( \rho (w, y) < 1/m \). It follows that \( \rho (x, y) < 2/m \), and condition (b) shows that \( y \in V^n_q \). As \( F_q \) meets \( V^n_p \), we have \( F_q \subseteq V^n_p \).

Next we deduce \( U^n_q \subseteq U^n_p \). The preceding argument has shown that \( F_q \subseteq U^n_p \cup \text{Int} (F_q) \); hence \( F_q \subseteq N^n_q \). Let \( z \) be any point of \( U^n_q \); then there exists \( y' \in F_q \) such that \( \rho (z, y') < 1/m < 1/2n \), from (a). Since \( y' \in N^n_q \), there exists \( x' \in \text{Fr} (F_q) \) such that \( \rho (x', y') < 1/2n \). Thus \( \rho (x', z) < 1/n \), so that \( z \in N^n_p \subseteq U^n_q \). This proves \( U^n_q \subseteq U^n_p \) if \( p \neq q \); but if \( p = q \) then \( U^n_q \subseteq U^n_p \) trivially. Hence \( V^n_q \supseteq V^n_p \) in any case, whence \( W_m(q) \subseteq W_m(p) \), and the proof is complete.

**Corollary 1.** If the conditions of the theorem are satisfied, and if \( S \) is separable, or locally separable, or compact, or locally compact, then \( E \) also has the corresponding property.

Compactness and separability are preserved by arbitrary continuous mappings. Suppose \( S \) locally compact; we prove \( E \) locally compact at \( p \in E \). Since \( \text{Fr} (F_p) \) is compact, it is covered by a finite number of open sets with compact closures. Thus, if \( n \) is large enough, \( N^n_p \) has a compact closure. So therefore have \( V^n_q \cap N^n_p \) and (since \( f \) is closed) \( f(V^n_q \cap N^n_p) \). But, if \( \text{Fr} (F_p) \neq \emptyset \), \( f(V^n_q \cap N^n_p) = f(V^n_q) = W_n(p) \).
If \( \text{Fr} (F_p) \neq \emptyset \), \( F_p \) is an open inverse set, so \( (p) \) is a compact neighborhood of \( p \). Finally, the proof for local separability is entirely similar.

**Remark.** Corollary 1 does not apply to local peripheral compactness.

**Corollary 2.** If \( f \) is a closed continuous mapping of a metric space

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6 A space is locally peripherally compact if every point has arbitrarily small neighborhoods with compact frontiers. Every 0-dimensional space is locally peripherally compact.
S onto a locally countably compact space \( E \), then \( E \) is metrizable (and consequently the sets \( \text{Fr}(f^{-1}(p)) \) are compact).

For each point \( p \) of \( E \) has a sequence of open neighborhoods \( X_n(p) = \{ q \mid F_q \subset S(F_p, 1/n) \} \) with the property \( \bigcap_n X_n(p) = \{ p \} \). Since \( E \) is locally countably compact and regular,\(^7\) one easily derives open neighborhoods \( Y_n(p) \) of \( p \) such that \( \text{Cl}(Y_n(p)) \subset X_n(p) \), \( Y_n(p) \supset Y_{n+1}(p) \), and \( \text{Cl}(Y_1(p)) \) is countably compact; here \( \text{Cl} \) denotes the closure. It is a straightforward matter to verify that the sets \( Y_n(p) \) form a basis of neighborhoods at \( p \); hence \( E \) satisfies the first axiom of countability, and Theorem 1 applies.

3. Monotone mappings. The condition that \( f \) be closed cannot be omitted in Theorem 1, in general \([2, \text{p. 70, Example 2}]\), even if \( f \) is open, \( S \) a subset of the plane, \( E \) a Hausdorff space with a countable basis, and the sets \( f^{-1}(p) \) are all compact. However, in some cases the closedness of \( f \) follows from the other hypotheses. This is so, for example, if \( S \) is compact and \( E \) is Hausdorff. A less obvious example is the following, which generalizes a theorem of A. V. Martin \([5, \text{Theorem 5}]\).

Theorem 2. Let \( E \) be the hyperspace of a decomposition of a locally peripherally compact metric space \( S \) into connected sets with compact frontiers; equivalently, let \( E = f(S) \) where \( f \) is a quasicompact continuous mapping such that, for each \( p \in E \), \( f^{-1}(p) \) is connected and \( \text{Fr}(f^{-1}(p)) \) is compact. Then, if \( E \) is a Hausdorff space, \( f \) is closed (and consequently \( E \) is metrizable, by Theorem 1); further, \( E \) is then locally peripherally compact.

As before, we write \( F_p = f^{-1}(p) \); we first prove that the decomposition \( \{ F_p \} \) of \( S \) is upper semi-continuous. Given any open set \( U \supset F_p \), cover \( \text{Fr}(F_p) \) by a finite number of open sets \( U_1, U_2, \ldots, U_m \) such that \( U_i \subset U \) and \( \text{Fr}(U_i) \) is compact \((1 \leq i \leq m)\). Let \( V = U_1 \cup U_2 \cup \cdots \cup U_m \cup \text{Int}(F_p) \); then \( F_p \subset V \subset U \), and the set \( \text{Fr}(V) \), being a closed subset of \( \text{Fr}(U_1) \cup \cdots \cup \text{Fr}(U_m) \cup \text{Fr}(F_p) \), is compact. Take \( W_n = S(F_p, 1/n) \cup \text{Int}(F_p) \), an open set containing \( F_p \); it is enough to prove that, for some \( n \), every \( F_q \) meeting \( W_n \) is contained in \( V \). Suppose this false; then, for each \( n \), we obtain \( q_n \in E \) such that \( F_{q_n} \) meets both \( W_n \) and \( S - V \). Since \( q_n \neq p \) (for \( F_p \subset V \)), \( F_{q_n} \) contains a point \( y_n \) of \( S(F_p, 1/n) \), and there exists \( x_n \in \text{Fr}(F_p) \) such that \( \rho(x_n, y_n) < 1/n \). Again, for all \( n \) greater than some \( n_0 \), we have \( 1/n < \rho(\text{Fr}(F_p), S - V) \), so that \( W_n \subset V \). Hence \( F_{q_n} \) meets both \( V \) and \( S - V \), and so meets \( \text{Fr}(V) \), say in \( z_n \) \((n > n_0)\).

For a suitable subsequence of values \( n' \) of \( n \), we have \( z_n' \rightarrow z \in \text{Fr}(V) \),

\(^7\) In fact, any closed continuous image of a normal space is normal \([10]\).
Now \( x_n \rightarrow x \in \text{Fr}(F_p) \), and therefore \( y_n \rightarrow x \). Now \( z \in F_p \), since \( V \) contains \( F_p \) and is disjoint from \( \text{Fr}(V) \). Let \( f(z) = q \); then \( q \in E - (p) \) and \( z \in F_q \). Since \( E \) is a Hausdorff space, there exist disjoint open inverse sets \( Y \supset F_p, Z \supset F_q \). If \( n' \) is large enough, \( y_n \in Y \) and \( z_n \in Z \); thus \( F_{y_n} \) meets, and so is contained in, both \( Y \) and \( Z \), giving a contradiction.

This proves the decomposition upper semi-continuous; the mapping \( f \) is therefore closed, and Theorem 1 applies. Finally, to prove \( E \) locally peripherally compact, suppose \( p \in E \) and an open set \( G \ni p \); as above, we construct an open set \( V \supset F_p \) such that \( V \subset f^{-1}(G) \) and \( \text{Fr}(V) \) is compact. Let \( W = \{ q \mid F_q \subset V \} = E - f(S - V) \), an open set such that \( p \in W \subset G \). Write \( X = f^{-1}(W) \); clearly \( X \subset V \). Suppose \( q \) is any point of \( \text{Fr}(W) \); then \( q \in W \), so \( F_q \) meets \( S - V \). But \( q \in W = \text{Cl}(f(X)) = f(X) \) (since \( f \) is closed), so \( q \in f(V) \) and \( F_q \) also meets \( V \). Being connected, \( F_q \) meets \( \text{Fr}(V) \); hence \( q \in f(\text{Fr}(V)) \). This proves \( \text{Fr}(W) \subset f(\text{Fr}(V)) \), which is compact; thus \( \text{Fr}(W) \) is also compact, q.e.d.

4. General quasicompact mappings. The condition that the sets \( F_p \) are all connected cannot be omitted from Theorem 2, even if they are all compact and \( S \) is a locally compact subset of the plane; examples to show this are easily constructed. We have, however, the following theorem (which generalizes another theorem of Martin [5, Corollary to Theorem 4]), in which the mapping \( f \) can in fact be neither open nor closed.

**Theorem 3.** If \( f \) is a quasicompact continuous mapping of a locally compact separable metric space \( S \) onto a Hausdorff space \( E \) with a locally countable basis, then \( E \) is a locally compact separable metric space.

**Lemma 1.** Let \( f \) be a quasicompact continuous mapping of a topological space \( S \) onto a Hausdorff space \( E \) with a locally countable basis at \( p \in E \), and let \( \{ U_n \} \) \( (n = 1, 2, \ldots) \) be an increasing sequence of open sets of \( S \) such that \( \bigcup U_n \cap F_p = f^{-1}(p) \). Then, for some \( n \), \( p \in \text{Int}(f(U_n)) \).

Let \( \{ W_n \} \) \( (n = 1, 2, \ldots) \) be a basis of neighborhoods of \( p \); we may suppose \( W_1 \supset W_2 \supset \ldots \). Without loss of generality, we may assume that each \( U_n \) meets \( F_p \). We show that, for some \( n, f(U_n) \supset W_n \). Suppose not; then, for each \( n \), there is a point \( q_n \in W_n - f(U_n) \). Write \( Q = \{ q_n \} \) \( (n = 1, 2, \ldots) \), \( X = f^{-1}(Q) \); as \( p \in \text{Cl}Q, Q \) is not closed, and therefore \( X \) is not closed and there exists a point \( x \in \text{Fr}(X) \). Then \( f(x) \in \text{Fr}(Q) \), and it is easy to deduce (from the fact that \( E \) is Hausdorff and \( q_n \rightarrow p \)) that \( f(x) = p \). Hence \( x \in F_p \), so \( x \in U_n \) for some \( n \). Then \( U_n - \bigcup \{ f^{-1}(q_n), 1 \leq m \leq n \} \) is an open set containing \( x \) but
disjoint from $X$, contradicting $x \in \overline{X}$ and establishing the lemma.

**Lemma 2.** If $E$ is the hyperspace of a decomposition $\{F_p\}$ of a locally compact topological space $S$, for which each $\text{Fr}(F_p)$ is $\sigma$-compact, and if $E$ is Hausdorff and has a locally countable base, then $E$ is locally compact.

As usual, we use $f$ to denote the natural quasicompact mapping of $S$ onto $E$, so that $f^{-1}(p) = F_p$. By hypothesis, given $p \in E$, we can write $\text{Fr}(F_p) = \bigcup K_n$ ($n = 1, 2, \ldots$) where $K_n$ is compact. Since $S$ is locally compact, $K_n$ can be covered by finitely many open sets with compact closures; in this way we obtain open sets $G_n \supseteq K_n$ such that $\text{Cl}(G_n)$ is compact and $G_1 \subset G_2 \subset \cdots \cdots$. Applying Lemma 1 to $U_n = G_n \cup f^{-1}(F_p)$, we can $p \in f^{-1}(f(U_n))$ for some $n$. If $\text{Fr}(F_p) \neq \emptyset$, this gives $p \in f^{-1}(f(G_n)) \subset f(\text{Cl}(G_n))$, which is compact and closed in the Hausdorff space $E$. If $\text{Fr}(F_p) = \emptyset$, $(p)$ is a compact neighborhood of $p$.

**Lemma 3.** If $E$ is the hyperspace of a decomposition $\{F_p\}$ of a topological space $S$ with a countable base, and if $E$ is Hausdorff and has a locally countable base, then $E$ has a countable base.

(For upper semi-continuous decompositions, this was proved by Whyburn [10, Theorem 10].)

Let $\mathcal{B} = \{B_m\}$ be a countable base of open sets of $S$; we prove $E$ has a base of the form $\{f^{-1}(f(U))\}$ where $U$ is a finite union of sets of $\mathcal{B}$. Given any $p \in E$ and any open set $G \ni p$, we have $F_p \subset f^{-1}(G)$, and can cover $F_p$ by a sequence of sets $B_{m_1}, B_{m_2}, \ldots$, of $\mathcal{B}$, all satisfying $B_{m_i} \subset f^{-1}(G)$. Write $U_n = B_{m_1} \cup \cdots \cup B_{m_n}$; by Lemma 1 we have $p \in f^{-1}(f(U_n))$ for some $n$, where $f(U_n) \subset G$, and the lemma is proved.

**Proof of Theorem 3.** As $S$ is here separable metric and locally compact, the same is true of each $\text{Fr}(F_p)$, which is therefore $\sigma$-compact. By Lemma 2, $E$ is locally compact, and therefore regular. Also $E$ has a countable base, by Lemma 3, and so it is metrizable.

As another immediate consequence of Lemma 3, we have:

**Corollary.** If $f$ is a quasicompact continuous mapping of a separable metric space $S$ onto a regular space $E$ with a locally countable base, then $E$ is separable metric.

The extension of Theorem 3 (and its corollary) to nonseparable spaces $S$ presents unexpected difficulties. We shall see from examples (§6) that even an open continuous image of a locally compact metric space need not be metrizable, even if it is regular (and has a locally countable base) and the sets $\text{Fr}(F_p)$ are compact. One fairly im-

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8 A set is $\sigma$-compact if it is the union of countably many compact sets.
mediate extension of Theorem 3 is as follows.

**Theorem 3'.** Let \( f \) be a quasicompact continuous mapping of a locally compact metric space \( S \) onto a Hausdorff space \( E \) with a locally countable base, and suppose that the sets \( Fr (f^{-1}(p)) \) are separable and that \( E \) is paracompact. Then \( E \) is a locally compact metric space.

For the argument proving Theorem 3 here shows that each \( p \in E \) has a compact neighborhood of the form \( f(G) \) where \( G \) is compact. Since \( E \) is Hausdorff, the mapping \( f \) restricted to \( G \) is closed; hence, by a well-known special case of Theorem 1, \( f(G) \) is metrizable. Thus \( E \) is locally metrizable. Since \( E \) is paracompact, \( E \) has a locally finite covering by closed metrizable sets, and is therefore [6] metrizable.

The condition in Theorem 3' that \( E \) be paracompact cannot be dispensed with, in general, even if \( f \) is open (§6). I do not know whether it is superfluous if the sets \( f^{-1}(p) \) are assumed to be separable. If \( f \) is open and the sets \( f^{-1}(p) \) are separable, the paracompactness of \( E \) follows from the other hypotheses (see the corollary to Theorem 4 below).

5. **Open mappings.** If \( f \) is open, Theorem 3 has the following extension to not necessarily separable spaces; perhaps the most interesting feature of the theorem is that none of the hypotheses is superfluous (§6).

**Theorem 4.** If \( f \) is an open continuous mapping of a metric, locally separable space \( S \) onto a regular space \( E \), and if for each \( p \in E \) the set \( f^{-1}(p) \) is separable, then \( E \) is metrizable and locally separable.

**Lemma 1.** The theorem is true if \( S \) is separable, and \( E \) is then separable.

This is a special case of the corollary to Theorem 3; it is also obvious directly.

**Lemma 2.** If \( f(S)=E \), where \( f \) is open and continuous and each \( f^{-1}(p) \) is separable, then for every separable subset \( Y \) of \( E \), \( f^{-1}(Y) \) is separable.

Let \( Q = \{ q_m \} \ (m = 1, 2, \ldots) \) be a countable subset of \( Y \) such that \( \overline{Q} \supset Y \), and for each \( m \) let \( X_m \) be a countable dense subset of \( f^{-1}(q_m) \). Write \( X = \bigcup X_m \); it is easy to see that the countable set \( X \) is dense in \( f^{-1}(Y) \).

**Proof of Theorem 4.** By a theorem of Alexandroff [1], \( S \) can be expressed as a union of pairwise disjoint nonempty open sets \( S_\lambda \), each of which is separable. Write \( S_\lambda \sim S_\mu \) to mean that there exists a finite sequence \( \lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu \) such that each set \( f(S_{\lambda_{i-1}}) \) meets
It is easily verified that \( \sim \) is an equivalence relation. Let the union of the \( S_\mu \)'s equivalent to \( S_\lambda \) be \( T_\lambda \); thus \( T_\lambda \) is open, and \( T_\lambda \) and \( T_\mu \) are either identical or disjoint. Further, \( T_\lambda \) is an inverse set, (i.e., equals \( f^{-1}(f(T_\lambda)) \)). For suppose \( F_\mu = f^{-1}(p) \) meets \( T_\lambda \), say in \( y \in F_\mu \cap S_\eta \) where \( S_\eta \sim S_\lambda \). If \( x \) is any point of \( F_\mu \), we have \( x \in S_\eta \) and \( p \in f(S_\eta) \cap f(S_\mu) \); hence \( S_\eta \sim S_\mu \sim S_\lambda \), so that \( x \in T_\lambda \). Thus \( F_\mu \subset T_\lambda \) whenever \( F_\mu \) meets \( T_\lambda \), and \( T_\lambda \) is an inverse set. It follows that the distinct sets \( f(T_\lambda) \) are disjoint and open, and they cover \( E \); to prove the theorem, it will suffice to prove that each \( f(T_\lambda) \) is separable metric, and by Lemma 1 it suffices to prove that each \( F_\mu \) is separable.

Now let \( T_\alpha^n \) denote the union of those sets \( S_\mu \) which can be reached from \( S_\lambda \) in at most \( n \) steps—that is, for which there is a sequence \( \lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu \) of the type used to define \( \sim \), with \( k \leq n \). Clearly \( T_\lambda = T_\alpha^n \cup T_\alpha^{n+1} \cup \cdots \), and it is enough to prove that each \( T_\alpha^n \) is separable. Suppose this true for one particular value of \( n \). Then \( T_\alpha^{n+1} \) consists of \( T_\alpha^n \) together with those sets \( S_\mu \) for which \( f(S_\mu) \) meets \( f(T_\alpha^n) \)—that is, for which \( S_\mu \) meets \( f^{-1}(f(T_\alpha^n)) \). By hypothesis, \( T_\alpha^n \) is separable; hence so is \( f(T_\alpha^n) \), and Lemma 2 now shows that \( f^{-1}(f(T_\alpha^n)) \) is also separable. It can therefore meet at most countably many of the disjoint open sets \( S_\mu \). Thus \( T_\alpha^{n+1} \) is a union of countably many separable sets, and is again separable. Since \( T_\alpha^n = S_\lambda \) is separable, the separability of \( T_\alpha^n \) follows for all \( n \); and the theorem is proved.

**Corollary.** If \( f \) is an open continuous mapping of a locally compact metric space \( S \) onto a Hausdorff space \( E \), and if for each \( p \in E \) the set \( f^{-1}(p) \) is separable, then \( E \) is a locally compact metric space. (Compare Theorem 3'.)

For, from the second lemma to Theorem 3, \( E \) is locally compact, and therefore regular; now Theorem 4 applies.

6. **Counterexamples.** The following examples show the need for the restrictions imposed in Theorems 3, 3' and 4.\(^9\) In each case, \( f \) will be an open (and so certainly quasi-compact) continuous mapping of a metric space \( S \) onto a nonmetrizable space \( E \); the openness of \( f \) is easily verified from the fact that \( E \) is in each case the hyperspace of a decomposition \( \{ F_\mu \} \) of \( S \) which is lower semi-continuous (that is, given any open set \( U \) meeting \( F_\mu \), there exists an open set \( V \supset F_\mu \) such that every \( F_\mu \) meeting \( V \) meets \( U \)). In the first two examples, \( S \) is locally compact; in all three, the openness of \( f \) guarantees that

\(^9\) See [2, p. 70, Ex. 1, 2] for examples showing that “Hausdorff” cannot be replaced by “\( T_1 \)” in Theorems 3 and 3’ (even if \( S \) is compact), and that “regular” cannot be replaced by “Hausdorff” in Theorem 4 (even if \( S \) is separable).
$E$ will have a locally countable base.

(1) For each countable ordinal $\alpha$, let $E_\alpha$ denote the set of ordinals not exceeding $\alpha$; in the usual topology, $E_\alpha$ is a compact metric space, and we suppose it metrized with diameter $<1$, say with metric $\rho_\alpha$. Let $S$ be the set of all ordered pairs $(\alpha, \beta)$, where $\alpha, \beta$ are countable ordinals and $\alpha \geq \beta$. Define a distance $\rho$ on $S$ by: $\rho\{(\alpha, \beta), (\alpha', \beta')\} = \rho_\alpha(\beta, \beta')$ if $\alpha = \alpha'$, 1 otherwise. Thus $S$ is the discrete union of the spaces $E_\alpha$, and is locally compact (all sets of diameter less than 1 having compact closures). The mapping $f$ given by $f(\alpha, \beta) = \beta$ is an open continuous mapping of $S$ onto the space $E$ of all countable ordinals (in its usual topology); the corresponding decomposition is $\{F_\beta\}$ where $F_\beta = \{(\alpha, \beta) | \alpha \geq \beta\}$. Here $E$ is a completely normal Hausdorff space, and is locally compact and has a locally countable base, but is not metrizable.

(2) Our second example is similar, but $E$ will satisfy fewer separation axioms; to compensate for this, the sets $F_p$ will have compact frontiers. Let $I$ denote the unit interval $0 \leq x \leq 1$, and let $A, R$ denote respectively the sets of irrational and rational numbers in $I$. For each $a \in A$, choose one sequence $\{r_n(a)\} \to a$ with $r_n(a) \in R$ (e.g., by using the decimal expansion). Now take $S = (R \times A) \cup A$ metrized as follows: $\rho\{(r_n(a), a), a\} = 1/m$, $\rho\{(r_n(a), a), (r_n(a), a)\} = |1/m - 1/n|$, and all other distances between distinct points are 1. Thus $S$ is the discrete union of sets $S_a = (R \times a) \cup a$, and $S_a$ consists of a sequence $\{(r_n(a), a)\}$ converging to $a$, together with a discrete sequence of points (the other points $(r, a)$). Clearly $S$ is a locally compact metric space (all closed sets of diameter less than 1 being compact). Now map $S$ in $I$ by: $f((r, a)) = r$, $f(a) = a$. The corresponding decomposition is characterized by $F_p = \{(r, a) | a \in A\}$, $F_a = \{a\}$. Each $F_p, (p \in I)$ is both open and closed in $S$, so in any case $Fr(F_p)$ is compact ($p \in I$) and has, in fact, at most one point. Further, from the fact that each $F_p$ is either open or consists of a single point, it is immediate that the decomposition $\{F_p\}$ is lower semi-continuous; thus $f$ is an open continuous mapping of $S$ onto the hyperspace $E$ of the decomposition, which is easily seen to consist of $I$ with the following topology. A typical neighborhood of $r \in R$ is $(r)$; a typical neighborhood of $a \in A$ consists of $a$ together with all points $r_n(a)$ for $n$ sufficiently large. These neighborhoods are closed, as well as open, in $E$; hence $E$ is a regular Hausdorff space. But $E$ is separable ($R$ is a countable dense set), yet contains an uncountable discrete set ($A$); hence $E$ is not metrizable. (If the continuum hypothesis is true, $E$ cannot be normal; see F. B. Jones, Concerning normal and completely normal spaces, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 671–677.)

(3) In our last example, the sets $F_p$ will themselves be compact,
though $S$ will no longer be locally separable. Let $N =$ set of all positive integers, $G =$ set of all mappings $g$ of $N$ in $N$. Roughly, $S$ will consist of $N, G$, a sequence of points converging to each $g \in G$, and $c$ sequences of points (one for each $g \in G$) converging to each $n \in N$. Precisely, we take $S = G \cup (G \times N) \cup N \cup (N \times G \times N)$, and define a metric $\rho$ on $S$ as follows:

$$\rho \{g, (g, n)\} = \frac{1}{n}; \quad \rho \{(g, n), (g, m)\} = \frac{1}{n} + \frac{1}{m} \text{ if } n \neq m;$$

$$\rho \{n, (n, g, m)\} = \frac{1}{m}; \quad \rho \{(n, g, m), (n, g', m')\} = \frac{1}{m} + \frac{1}{m'} \text{ if } (g, m) \neq (g', m');$$

and all other distances between distinct points are 2. It is easily verified that $\rho$ is a metric. Note that the points $(g, m)$ and $(n, g, m)$ are isolated, each having distance at least $1/m$ from any other point. Decompose $S$ into the 2-point sets $F_p = \{(g, n), (n, g, g(n))\}$ and the other single points. The decomposition is lower semi-continuous (for the sets $F_p$ with more than one point are open), and so corresponds to an open continuous mapping $f$ of $S$ onto the hyperspace $E$; further the sets $F_p (= f^{-1}(p))$ are all compact (having, in fact, at most 2 points). It is a straightforward matter to verify that $E$ is a regular $T_1$ space. Now the sets $f(G), f(N)$ are disjoint and closed in $E$; if $E$ were normal, there would be disjoint open sets $U \supset f(N), V \supset f(G)$. Then $N \subset f^{-1}(U)$, which is open in $S$; hence, for each $n \in N$, we can choose $m = h(n)$ (say) large enough for $S(n, 1/(m-1)) \subset f^{-1}(U)$, and this gives $f(n, g, h(n)) \in U$. This is true for all $g \in G$, and so in particular for $g = h$; thus $f(h, n) = f(n, h, h(n)) \in U$, for all $n \in N$. But if $n$ is large enough, $(h, n) \in S(h, 1/(n-1)) \subset f^{-1}(V)$, so that $f(h, n) \in V$, contradicting $U \cap V = \emptyset$. Thus $E$ is not normal, and so certainly not metrizable.10

7. Open-closed mappings. We conclude by considering the case in which the mapping $f$ of the metric space $S$ is simultaneously open and closed—or, equivalently, in which the decomposition $\{F_p\}$ of $S$ is continuous (i.e., both upper and lower semi-continuous). The essential content of the following theorem is due to Wallace [8], but Theorem 1 makes the proof very simple.

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10 Incidentally, example (3) answers a question raised in [2, p. 70] by giving a decomposition of a normal space for which the hyperspace is regular but not normal. Another example answering this question has been given by T. Ganea, On the Prüfer manifold and a problem of Alexandroff and Hopf, Acta Scientiarum Math. vol. 15 (1954) pp. 231–235.
Theorem 5. Necessary and sufficient conditions for a decomposition of a metric space \( S \) into disjoint nonempty closed sets \( F_p \) to be continuous are:

(a) Each \( F_p \) is either open or compact.

(b) Given a compact \( F_p \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that whenever \( \rho(F_p, F_q) < \delta \) we have \( d(F_p, F_q) < \epsilon \) (\( d \) denoting the Hausdorff distance).

To prove (a) and (b) necessary, we note that the decomposition space \( E \) now has a locally countable basis (for the corresponding mapping \( f \) of \( S \) onto \( E \) is open), so by Theorem 1 each \( F_p \) is compact. If \( \text{Int}(F_p) = \emptyset \), this proves \( F_p \) compact. If \( \text{Int}(F_p) \neq \emptyset \), \( f(\text{Int}(F_p)) = (p) \) is open in \( E \), so \( F_p \) is open in \( S \). Thus (a) follows; and (b) is now a routine restatement of upper and lower semi-continuity. Conversely, if (a) and (b) are given, it is easy to deduce that the decomposition is both upper and lower semi-continuous.

Corollary. If \( f(S) = E \), where \( f \) is an open, closed, continuous mapping of a metric space \( S \), then \( E \) is metrizable, with metric \( \sigma \) given by \( \sigma(p, q) = d(f^{-1}(p), f^{-1}(q)) \).

References


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\footnote{This corollary is due to V. K. Balanchandran, A mapping theorem for metric spaces, Duke Math. J. vol. 22 (1955) pp. 461–464.}