# AN IDENTITY FOR $l$-GROUPS[1]

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G. Birkhoff [1, p. 109, Theorem 7.8] discovered that[2]

$$(1) \qquad |a \cup c - b \cup c| + |a \cap c - b \cap c| = |a - b|$$

for all $a$, $b$ and $c$ in any vector lattice. In [2, p. 309] he stated that he had been unable to generalize the identity (1) to noncommutative $l$-groups. However he proved (cf. [3, p. 220, Theorem 8(20)]) that

$$(2) \qquad |a \cup c - b \cup c| \leqq |a - b|, \text{ and dually,}$$

for all $a$, $b$ and $c$ in any $l$-group.

We shall prove that the identity (1) actually does hold in all $l$-groups. Clearly, this gives a new proof of (2).

We begin by proving that, for all $a$, $b$ and $c$ in any $l$-group,

$$(3) \quad (a \cap b) \cup c - c + (a \cup b) \cap c = (a \cup b) \cap c - c + (a \cap b) \cup c.$$

Indeed,

$$\begin{aligned}(a \cap b) \cup (b \cap c) \cup (c \cap a) &= (a \cap b) \cup \{(a \cup b) \cap c\} \\ &= a \cap b - (a \cap b) \cap (a \cup b) \cap c + (a \cup b) \cap c \\ &= a \cap b - (a \cap b) \cap c + (a \cup b) \cap c \\ &= a \cap b - a \cap b + (a \cap b) \cup c - c + (a \cup b) \cap c;\end{aligned}$$

thus

$$(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cap b) \cup c - c + (a \cup b) \cap c.$$

Dually,

$$(a \cup b) \cap (b \cup c) \cap (c \cup a) = (a \cup b) \cap c - c + (a \cap b) \cup c.$$

Since [3, p. 133] $(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cup b) \cap (b \cup c) \cap (c \cup a)$, this proves (3).

We now prove the identity (1). For all $a$, $b$, and $c$ in any $l$-group, we have

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[1] The result of this paper is taken from the author's doctoral thesis (Harvard University, 1955). The author now holds a Research Fund Fellowship from the University of New Zealand.

[2] We make the convention that the lattice operations in an $l$-group have priority over the group operations: for example, $a - b \cap c$ means $a - (b \cap c)$.

$$
\begin{aligned}
|a \cup c - b \cup c| &+ |a \cap c - b \cap c| \\
&= (a \cup c) \cup (b \cup c) - (a \cup c) \cap (b \cup c) + (a \cap c) \cup (b \cap c) \\
&\quad - (a \cap c) \cap (b \cap c) \\
&= (a \cup b) \cup c - (a \cap b) \cup c + (a \cup b) \cap c - (a \cap b) \cap c \\
&= a \cup b - (a \cup b) \cap c + c - (a \cap b) \cup c + (a \cup b) \cap c \\
&\quad - c + (a \cap b) \cup c - a \cap b \\
&= a \cup b - \{(a \cap b) \cup c - c + (a \cup b) \cap c\} \\
&\quad + \{(a \cup b) \cap c - c + (a \cap b) \cup c\} - a \cap b \\
&= a \cup b - a \cap b, \quad \text{by (3).}
\end{aligned}
$$

Since $a \cup b - a \cap b = |a - b|$, this completes the proof of (1).

## References

1. G. Birkhoff, *Lattice theory*, 1st ed., New York, 1940.

2. ———, *Lattice-ordered groups*, Ann. of Math. (2) vol. 43 (1942) pp. 298–331.

3. ———, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, 1940; rev. ed., New York, 1948.

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