AN IDENTITY FOR \( l \)-GROUPS

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G. Birkhoff [1, p. 109, Theorem 7.8] discovered that
\[
|a \cup c - b \cup c| + |a \cap c - b \cap c| = |a - b|
\]
for all \( a, b \) and \( c \) in any vector lattice. In [2, p. 309] he stated that he had been unable to generalize the identity (1) to noncommutative \( l \)-groups. However, he proved (cf. [3, p. 220, Theorem 8(20)]) that
\[
|a \cup c - b \cup c| \leq |a - b|, \text{ and dually,}
\]
for all \( a, b \) and \( c \) in any \( l \)-group.

We shall prove that the identity (1) actually does hold in all \( l \)-groups. Clearly, this gives a new proof of (2).

We begin by proving that, for all \( a, b \) and \( c \) in any \( l \)-group,
\[
(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cap b) \cup (a \cup b) \cap c = (a \cap b) \cup c.
\]
Indeed,
\[
(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cap b) \cup \{(a \cup b) \cap c\}
\]
\[
= a \cap b - (a \cap b) \cap (a \cup b) \cap c + (a \cup b) \cap c
\]
\[
= a \cap b - (a \cap b) \cap c + (a \cup b) \cap c
\]
\[
= a \cap b - a \cap b + (a \cap b) \cup c - c + (a \cup b) \cap c;
\]
thus
\[
(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cap b) \cup c - c + (a \cup b) \cap c.
\]
Dually,
\[
(a \cup b) \cap (b \cup c) \cap (c \cup a) = (a \cup b) \cup c - c + (a \cap b) \cup c.
\]
Since [3, p. 133] \((a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cup b) \cap (b \cup c) \cap (c \cup a)\), this proves (3).

We now prove the identity (1). For all \( a, b \), and \( c \) in any \( l \)-group, we have

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\[\text{\footnote{We make the convention that the lattice operations in an \( l \)-group have priority over the group operations: for example, \( a - b \cap c \) means \( a - (b \cap c) \).}}\]

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\[ |a \cup c - b \cup c| + |a \cap c - b \cap c| \]
\[
= (a \cup c) \cup (b \cup c) - (a \cup c) \cap (b \cup c) + (a \cap c) \cup (b \cap c) \\
- (a \cap c) \cap (b \cap c)
\]
\[
= (a \cup b) \cup c - (a \cap b) \cup c + (a \cup b) \cap c - (a \cap b) \cap c
\]
\[
= a \cup b - (a \cap b) \cap c + c - (a \cap b) \cup c + (a \cup b) \cap c \\
- c + (a \cap b) \cup c - a \cap b
\]
\[
= a \cup b - \{(a \cap b) \cup c - c + (a \cup b) \cap c\} \\
+ \{(a \cup b) \cap c - c + (a \cap b) \cup c\} - a \cap b
\]
\[
= a \cup b - a \cap b, \text{ by (3).}
\]
Since \(a \cup b - a \cap b = |a - b|\), this completes the proof of (1).

REFERENCES


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