BOUNDARY VALUES OF CONTINUOUS ANALYTIC FUNCTIONS

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Let U, K, and C denote the open unit disc, the closed unit disc, and the unit circumference, respectively. Let A be the set of all complex-valued functions which are defined and continuous on K and analytic in U.

The theorem proved in this paper states that if E is a closed subset of C, of (Lebesgue) measure zero, and if ϕ is a continuous function defined on E, then there exists a function $f \in A$ which is an extension of ϕ, i.e., $f(z) = \phi(z)$ for all $z \in E$ (actually, a little more is proved about the way in which the range of f can be restricted). This theorem may be regarded as a strengthened form of a result due to Fatou [**3**, p. 393] which asserts that under the above assumptions on E there exists a function $f \in A$ which vanishes at every point of E and at no other point of K.

A well-known result due to F. and M. Riesz [**4**] shows that the theorem is false for any set E whose closure has positive measure. It is perhaps more interesting to observe that the conclusion of the theorem cannot be strengthened by asserting, for instance, that f satisfies a Lipschitz condition if ϕ does: if E consists of the points 1 and $\exp(i/\log n)$ $(n = 2, 3, 4, \cdots)$, and if $\phi(z) = 1/z$, then for any function $f \in A$ which extends ϕ, the function $g(z) = zf(z)$ is nonconstant ($g(0) = 0$) and $g(z) = 1$ for all $z \in E$; this function g cannot satisfy a Lipschitz condition [**1**, p. 13] and the same is true of f. Carleson's work [**2**] on this and related problems should be mentioned here.

A subset of the plane homeomorphic to K will be called a two-cell. For every two-cell T there exists a mapping which will be called a conformal mapping of K onto T; more precisely, this is a homeomorphism of K onto T which maps U conformally onto the interior of T.

The square with vertices at 0, 1, i will be denoted by S. For any real number t, tS is the set of all z of the form $z = tw$, $w \in S$.

A function is said to be simple if it has only a finite set of values.

THEOREM. *Suppose*

(a) *E is a closed subset of C, $m(E) = 0$;*

(b) *ϕ is a continuous complex valued function on E;*

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(c) *T is a two-cell such that $\phi(E) \subset T$.*
Then there exists a function $f \in A$ such that
(i) *$f(z) = \phi(z)$ for all $z \in E$;*
(ii) *$f(K) \subset T$.*

LEMMA 1. *If ϕ is a simple continuous function on E whose real part is non-negative, then there exists a function $f \in A$ such that* (i) *f is an extension of ϕ, and* (ii) *$R[f(z)] \geqq 0$ for all $z \in K$.*

PROOF. Since every simple continuous function is a finite linear combination of continuous characteristic functions, it suffices to consider the case in which $E = E_0 \cup E_1$ and $\phi(z) = 0$ on E_0, $\phi(z) = \alpha$ on E_1, with $R[\alpha] \geqq 0$, $\alpha \neq 0$.

For every closed set $H \subset C$ of measure zero, one can construct an integrable function $\mu > 1$ on C such that

(a) μ is continuous on $C - H$,
(b) if $z_0 \in H$, $\mu(z_0) = +\infty$ and $\mu(z) \to +\infty$ as $z \to z_0$,
(c) on any closed subarc of $C - H$, μ has a bounded derivative.

Properties (a) and (b) can be summarized by saying that μ is continuous on C, in the extended sense.

The Poisson integral of this function μ yields a function $u(H, z)$, defined on K, which is continuous on K in the extended sense, and which has the value $+\infty$ at every point of H (and nowhere else); the conjugate harmonic function $v(H, z)$ is continuous on $K - H$ [3, pp. 342–344, 360–361]. Put

$$g(H, z) = u(H, z) + iv(H, z) \qquad (z \in K)$$

with the understanding that $g(H, z) = \infty$ if $z \in H$. The values of g lie in the half-plane $u > 1$, so that g has a single-valued square root $h(H, z)$, defined on K, whose values lie in the domain $|\arg h| < \pi/4$, $R[h] > 1$; note that $h(H, z) = \infty$ if and only if $z \in H$.

It is now easy to verify that the function

$$q(z) = \frac{h(E_1, z)}{h(E_0, z) + h(E_1, z)} \qquad (z \in K)$$

is a member of A, that $q(z) = 0$ if and only if $z \in E_0$, $q(z) = 1$ if and only if $z \in E_1$, and that $0 \leqq R[q(z)] \leqq 1$ for all $z \in K$. Thus $q(K)$ is contained in a closed rectangle M which lies in the closed right half-plane and has 0 and 1 as boundary points. Let M_1 be another such rectangle, with 0 and α as boundary points, and let ψ be a conformal mapping of M onto M_1 which satisfies the conditions $\psi(0) = 0$, $\psi(1) = \alpha$. Then $f(z) = \psi(q(z))$ is the desired function.

LEMMA 2. *If ϕ is a simple continuous function on E which maps E* *into a two-cell T, then there exists a function $f\in A$ which is an extension of ϕ and which maps K into T.*

PROOF. Let z_0 be a boundary point of T which is not a value of ϕ, and let ψ map the right half-plane conformally onto the interior of T, such that $\psi(\infty)=z_0$. By Lemma 1, there is a function $g\in A$ with non-negative real part, which coincides with $\psi^{-1}(\phi(z))$ for all $z\in E$; thus the function $f(z)=\psi(g(z))$ has the desired properties.

LEMMA 3. *If ϕ is a continuous function on E which maps E into S, then there exists a sequence $\{\phi_n\}$ of simple continuous functions on E such that*

$$\text{(i)} \qquad \phi(z) = \sum_{n=1}^{\infty} \phi_n(z) \qquad (z \in E),$$

$$\text{(ii)} \qquad \phi_n(E) \subset 2^{-n}S \qquad (n = 1, 2, 3, \cdots).$$

PROOF. Let $\phi_0(z)=0$ and suppose $\phi_0, \cdots, \phi_{n-1}$ have been defined for some $n\geqq 1$, such that

$$\lambda_{n-1}(E) \subset 2^{-n+1}S,$$

where $\lambda_{n-1}=\phi-\phi_0- \cdots -\phi_{n-1}$. Since E is totally disconnected, E is the union of disjoint closed sets $E_1, \cdots, E_p$ such that the oscillation of λ_{n-1} on E_k is less than 2^{-n}. That is to say, there are squares $Q_1, \cdots, Q_p$ with edges of length 2^{-n} and parallel to the axes, such that

$$\lambda_{n-1}(E_k) \subset Q_k \subset 2^{-n+1}S \qquad (1 \leqq k \leqq p).$$

It is clear that Q_k and $2^{-n}S$ have a point c_k in common. Define a function ϕ_n on E by

$$\phi_n(z) = c_k \qquad (z \in E_k,\ 1 \leqq k \leqq p).$$

If $\lambda_n=\lambda_{n-1}-\phi_n=\phi-\phi_0- \cdots -\phi_n$, then $\lambda_n(E)\subset 2^{-n}S$, and the process continues. This proves the lemma.

PROOF OF THE THEOREM. Suppose first that $T=S$, and let ϕ_n be the functions of Lemma 3. By Lemma 2 there are functions $g_n\in A$ which are extensions of ϕ_n and which map K into $2^{-n}S$ $(n=1, 2, 3, \cdots)$. Define

$$f(z) = \sum_{n=1}^{\infty} g_n(z) \qquad (z \in K).$$

The fact that $g_n(K)\subset 2^{-n}S$ for $n=1, 2, 3, \cdots$ implies first of all that the series converges uniformly, so that $f\in A$, and secondly that $f(K)\subset S$; for $z\in E$,

$$f(z) = \sum_{n=1}^{\infty} g_n(z) = \sum_{n=1}^{\infty} \phi_n(z) = \phi(z).$$

The general case follows by an application of the Riemann mapping theorem, as in the proof of Lemma 2.

AN APPLICATION (*Added* July 31, 1956). An algebra R of continuous complex-valued functions on K is called a *maximum modulus algebra* if to every $f \in R$ there is a point $z_0 \in C$ such that $\max_{z \in K} |f(z)| = |f(z_0)|$. In [5] the following is proved: If R is a maximum modulus algebra which contains (i) a function which is one-to-one, (ii) a nonconstant function which is analytic in U, then every member of R is analytic in U. It is now possible to show that (i) cannot be omitted from the hypotheses.

To do this, consider the bicylinder $K \times K$ (in the space of two complex variables) whose distinguished boundary is the Cartesian product $C \times C$. Let E be a perfect subset of C, of measure zero. There exist continuous functions ϕ and ψ, defined on E, such that the mapping

$$z \rightarrow (\phi(z), \psi(z))$$

maps E *onto* $C \times C$. The theorem proved in the present paper shows that there is a function f, continuous on K, analytic in U, such that $f(K) \subset K$ and $f(z) = \phi(z)$ on E. Let g be a function which is continuous on K, *not* analytic in U, such that $g(K) \subset K$ and $g(z) = \psi(z)$ on E; we could take g to be conjugate-analytic in U.

Since every function analytic on $K \times K$ attains its maximum modulus on $C \times C$, it is clear that every member of the algebra generated by f and g attains its maximum modulus on E. We have thus constructed a maximum modulus algebra which contains both analytic and nonanalytic functions; it is to be noted, however, that this algebra does not separate points on K.

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