LIMIT-PRESERVING EMBEDDINGS OF PARTIALLY ORDERED SETS IN DIRECTED SETS

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Let \((A, >)\) be a partially ordered set. Then we say \([3]\) that a function \(f\) on \(A\) decides for \(S\) (on \(A\)) if \(\{\alpha | f(\alpha) \in S\}\) contains a cofinal residual subset of \(A\), i.e., if \(\gamma \in A\), there is a \(\beta \geq \gamma\) such that for all \(\alpha \geq \beta\), \(f(\alpha) \in S\). Clearly the intersection of any two cofinal residual sets is cofinal residual. Ginsburg \([2]\) showed that any partially ordered set \((A, >)\) can be embedded in an everywhere branching set \((B, >)\) in such a manner that the "natural" extension of \(f\) to \(B\) decides for the same sets as \(f\) does. Day \([1]\) has an embedding in a directed set preserving decision, but possibly creating new decisions, even turning a nonconvergent function into a convergent one. We show here that the embedding can be made precise. First, let us notice that

**Lemma.** Let \((A, >)\) and \((B, >)\) be partially ordered sets and let \(\phi\) map \(B\) into \(A\). If

(i) for every \(T\) cofinal residual in \(A\), \(\phi^{-1} T\) is cofinal residual in \(B\), and

(ii) for every \(U\) cofinal residual in \(B\), \(\phi U\) is cofinal residual in \(A\),

then for every \(f\) and \(S\), \(f\) decides for \(S\) on \(A\) if and only if \(f \phi\) decides for \(S\) on \(B\).

We may assume (by a trivial preliminary embedding) that the given set \(A\) has no least cofinal residual set, i.e., the set of maximal elements is not cofinal. Let \(B\) be the set of all pairs \((T, \alpha)\) such that \(\alpha \in T\) and \(T\) is cofinal residual in \(A\). We define \((T_1, \alpha_1) > (T_2, \alpha_2)\) if \(T_1 \subset T_2\) or \(T_1 = T_2\) and \(\alpha_1 > \alpha_2\). We define \(\phi(T, \alpha) = \alpha\). Thus we see that \((B, >)\) is directed and therefore cofinal residual reduces to residual. Now if we replace \((A, \alpha)\) by \(\alpha\), we see that

**Theorem.** Every partially ordered set \((A, >)\) can be embedded in a directed set \((B, >)\) in such a manner that for any function \(f\) on \(A\), the "natural" extension of \(f\) to \(B\) decides for precisely those sets for which \(f\) decides.

**References**


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THE ZEROS OF CERTAIN SINE-LIKE INTEGRALS

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We establish here the monotonic character of the zeros (modulo 1) of

\[ \int_{x}^{\infty} \frac{f(t)}{t} \, dt, \quad x > 0, \]

where \( f(t) \) satisfies the conditions

(C1). \( f(t) \geq 0 \) for \( 0 \leq t < 1 \);
(C2). \( f(t) \neq 0 \) on any subinterval of \( 0 \leq t < 1 \);
(C3). \( f(t+n) = (-1)^n f(t) \) for \( n = 1, 2, 3, \ldots \);
(C4). \( f(t)/t \) is Lebesgue integrable on \( 0 \leq t \leq 1 \).

It is clear that these conditions imply that the integral (1) has precisely one zero, say \( z_n \), in the interval \( n < x < n+1 \).

Let \( C \) be defined (uniquely) by the conditions

\[ 2 \int_{0}^{C} f(t) \, dt = \int_{0}^{1} f(t) \, dt, \quad 0 < C < 1. \]

Now,

(A) \( z_n - n \geq C \) for \( n = 0, 1, 2, \ldots \),
(B) \( z_n - n \to C \) as \( n \to \infty \),

as was shown in [2], even more generally, with the factor \( 1/t \) of \( f(t) \) in (1) replaced by a function denoted there by \( g(t) \) of which \( 1/t \) is a special case. When \( f(t) = \sin \pi t \), the sequence \( \{z_n - n\} \) is decreasing, as Harry Pollard has shown, and I. I. Hirschman has observed that Pollard's proof applies equally well to the zeros of

\[ \int_{x}^{\infty} g(t) \sin \pi t \, dt \]

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