PARTITIONS OF MULTI-PARTITE NUMBERS

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1. Introduction. In what follows all small latin letters denote non-negative rational integers. We suppose for the present that $|X_i| < 1$ ($1 \leq i \leq j$) and write

$$F_i(Y) = F_i(X_1, \cdots, X_j; Y) = \prod (1 + X_1^{k_1} \cdots X_j^{k_j} Y)$$

and

$$G_i(Y) = \{F_i(-Y)\}^{-1} = \prod (1 - X_1^{k_1} \cdots X_j^{k_j} Y)^{-1},$$

where the products extend over all non-negative $k_1, \cdots, k_j$. If $|Y| < 1$, we have

$$G_i(Y) = 1 + \sum_{n=1}^{\infty} Q_i(n) Y^n,$$

where

$$Q_i(n) = Q_i(X_1, \cdots, X_j; n) = \sum_{n_1, \cdots, n_j=0}^{\infty} q(n_1, \cdots, n_j; n) X_1^{n_1} \cdots X_j^{n_j}$$

and $q(n_1, \cdots, n_j; n)$ is the number of partitions of the $j$-partite number $(n_1, \cdots, n_j)$ into just $n$ parts, that is the number of solutions of the "vector" equation (or equation in single row matrices)

$$\sum_{k=1}^{n} (x_{1k}, \cdots, x_{jk}) = (n_1, \cdots, n_j).$$

The order of the vectors on the left-hand side of (1) is irrelevant. Again

$$F_i(Y) = 1 + \sum_{n=1}^{\infty} R_i(n) Y^n,$$

where

$$R_i(n) = \sum r(n_1, \cdots, n_j; n) X_1^{n_1} \cdots X_j^{n_j}$$

and $r(n_1, \cdots, n_j; n)$ is the number of partitions of $(n_1, \cdots, n_j)$ into just $n$ different parts, that is, the number of solutions of (1) in which the vectors on the left hand side are all different.

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If \( j = 1 \), we have
\[
(1 - Y)G_1(Y) = G_1(X_1Y)
\]
and so
\[
Q_1(n) - Q_1(n - 1) = X_1^n Q_1(n),
\]
whence
\[
Q_1(n) = \frac{Q_1(n - 1) + 1}{1 - X_1^n} = \frac{1}{(1 - X_1)(1 - X_1^2) \cdots (1 - X_1^n)}.
\]
Similarly we find that
\[
R_1(n) = \frac{X_1^{n(n-1)/2}}{(1 - X_1)(1 - X_1^2) \cdots (1 - X_1^n)}.
\]

Macmahon (*Combinatory analysis ii*, Cambridge, 1916) discussed in detail the case \( j = 1 \) and referred briefly to the more general case, commenting on its complexity. More recently Bellman (*Bull. Amer. Math. Soc. Research Problem 61-1-3*) has asked for a formula for \( Q_j(n) \). My object here is to obtain formulae for \( Q_j(n) \) and \( R_j(n) \) for general \( j \) and \( n \). For \( j > 1 \), these formulae cannot be reduced to anything as simple as in the case \( j = 1 \), but we can make some progress in this direction and deduce certain results about partitions.

2. The formulae for \( Q_j(n) \) and \( R_j(n) \). Let
\[
\alpha_1, \alpha_2, \alpha_3, \ldots
\]
be any infinite sequence such that \( |\alpha_k| < 1 \) for every \( k \) and \( \sum |\alpha_k| < \infty \). We write
\[
C(Y) = \prod_{k=1}^{\infty} (1 + \alpha_k Y) = 1 + \sum_{n=1}^{\infty} A(n)Y^n
\]
and
\[
D(Y) = \{C(-Y)\}^{-1} = \prod_{k=1}^{\infty} (1 + \alpha_k Y + \alpha_k^2 Y^2 + \cdots) = 1 + \sum_{n=1}^{\infty} B(n)Y^n.
\]
Clearly \( A(n) \) is the sum of the products of every set of \( n \) different \( \alpha \)
and \( B(n) \) is the sum of the products of every set of \( n \) numbers \( \alpha \),
repetitions permitted. We write also

\[ S(m) = \sum_k \alpha_k^m. \]

We see at once that

\[ \log D(Y) = -\sum_{k=1}^{\infty} \log (1 - \alpha_k Y) = \sum_{m=1}^{\infty} \frac{S(m)}{m} Y^m. \]

Hence

\[ D(Y) = \exp \left\{ \sum_{m=1}^{\infty} \frac{S(m)}{m} Y^m \right\} \]

and

\[ B(n) = \sum_{(n)} \prod \frac{\{S(m)\}^{h_m}}{h_m! m^{h_m}}, \]

the sum extending over all partitions of \( n \) of the form

\[ n = \sum h_m m \]

and the product over all the different parts \( m \) in the partition. Again

\[ C(-Y) = \exp \left\{ -\sum_{m=1}^{\infty} \frac{S(m)}{m} Y^m \right\} \]

and so

\[ A(n) = (-1)^n \sum_{(n)} \prod \frac{(-1)^{h_m} \{S(m)\}^{h_m}}{h_m! m^{h_m}}. \]

Next, if we differentiate (3) with respect to \( Y \) and multiply through by \( D(Y) \), we obtain

\[ \sum_{n=1}^{\infty} n B(n) Y^{n-1} = \sum_{m=1}^{\infty} S(m) Y^{n-1} \left\{ 1 + \sum_{n=1}^{\infty} B(n) Y^n \right\} \]

and so, equating coefficients of \( Y^{n-1} \), we have

\[ n B(n) = \sum_{m=1}^{n} S(m) B(n - m). \]

Similarly

\[ n A(n) = \sum_{m=1}^{n} (-1)^{m-1} S(m) A(n - m). \]
If we now take all of
\[ X_1^{k_1} \cdots X_j^{k_j} \quad (k_i \geq 0, 1 \leq i \leq j) \]
for the \( a \) in (2), we see that
\[ A(n) = R_j(n), \quad B(n) = Q_j(n) \]
and
\[ S(m) = \sum X_1^{m_1} \cdots X_j^{m_j} = \prod_{i=1}^{j} \left( \frac{1}{1 - X_i^m} \right) = \frac{1}{\beta_j(m)}, \]
where
\[ \beta_j(m) = \prod_{i=1}^{j} (1 - X_i^m). \]
Hence we have
\[ (6) \quad Q_j(n) = \sum_{(n)} \prod (h_m!)^{-1} \left\{ m \beta_j(m) \right\}^{-h_m} \]
and
\[ (7) \quad R_j(n) = (-1)^n \sum_{(n)} \prod (-1)^m (h_m!)^{-1} \left\{ m \beta_j(m) \right\}^{-h_m}. \]
These are the formulae for \( Q_j(n) \) and \( R_j(n) \). For \( j=1 \), they were found by Macmahon (loc. cit.).

Again, (4) and (5) become
\[ (8) \quad nQ_j(n) = \sum_{m=1}^{n} \frac{Q_j(n-m)}{\beta_j(m)} \]
and
\[ nR_j(n) = \sum_{m=1}^{n} (-1)^n \frac{R_j(n-m)}{\beta_j(m)} \]
If \( \sum h_m m = n \), it is easy to show that
\[ \frac{(1 - X)(1 - X^2) \cdots (1 - X^n)}{\prod (1 - X^m)^{h_m}} \]
is a polynomial in \( X \). Its degree is clearly
\[ \sum_{h=1}^{n} k - \sum h_m m = \frac{1}{2} n(n+1) - n = \frac{1}{2} n(n - 1). \]
Hence, if we write
we see from (6) that $P_j(n)$ is a polynomial of degree at most \( n(n-1)/2 \) in each of \( X_1, \ldots, X_j \).

It follows from its definition that \( Q_j(n) \) is a multiple infinite power series in \( X_1, \ldots, X_j \), the coefficient of each term being a non-negative integer. Since the \( \beta \) are polynomials with integral coefficients, we see that all the coefficients in the polynomial \( P_j(n) \) are integers. It seems very likely that all these coefficients are non-negative, but this I have not been able to prove. In §1, we saw that

\[
P_1(n) = 1
\]

for all \( n \). Unfortunately nothing so simple is true for \( j > 1 \).

3. Properties of \( P_j(n) \). We now suppose that \( Q_j(n) \) and \( R_j(n) \) are defined by (6) and (7), so that \( Q_j(n) \) and \( R_j(n) \) are rational functions defined for all values of the \( X_i \) except the \( m \)th roots of unity for which \( 1 \leq m \leq n \). Again, since \( P_j(n) \) is a polynomial, it can be defined for all values of the \( X_i \) without exception. We write \( P_j(0) = Q_j(0) = R_j(0) = 1 \) and see that \( P_j(1) = 1 \).

We have now

\[
\beta_j(X_1, \ldots, X_j, X_j^{-1}; m) = (1 - X_1^m) \cdots (1 - X_{j-1}^m)(1 - X_j^{-m}) = -X_j^{-m} \beta_j(X_1, \ldots, X_{j-1}, X_j^{-1}; m).
\]

Hence, by (6) and (7),

\[
Q_j(X_1, \ldots, X_j, X_j^{-1}; n) = X_j^n \sum_{(n)} (-1)^m (h_m!)^{-1} \{m\beta_j(m)\}^{-h_m}
\]

and

\[
R_j(X_1, \ldots, X_j, X_j^{-1}; n) = (-1)^n X_j^n R_j(X_1, \ldots, X_j; n).
\]

This transformation applies also with any one of \( X_1, \ldots, X_{j-1} \) in place of \( X_j \). Applying it twice, we have

\[
Q_j(X_1, \ldots, X_{j-2}, X_{j-1}, X_j^{-1}; n) = X_j^{n-1}X_j^n Q_j(X_1, \ldots, X_j; n).
\]

Using (9), we see that

\[
R_j(X_1, \ldots, X_j; n) = \frac{X_j^{n-1/2} P_j(X_1, \ldots, X_j, X_j^{-1}; n)}{\beta_j(1) \cdots \beta_j(n)},
\]
so that, if we can evaluate $P_j(n)$, we have a simple form for both $Q_j(n)$ and $R_j(n)$. Again

$$(X_{j-1}X_j)^{n(n-1)/2} P_j(X_1, \cdots, X_{j-2}, X_{j-1}^{-1}, X_j^{-1}; n) = P_j(X_1, \cdots, X_{j}; n).$$

If, then, we write

$$g = n(n-1)/2$$

and

$$P_j(X_1, \cdots, X_j; n) = \sum_{k_1, \cdots, k_j=0}^g \lambda(k_1, \cdots, k_j) X_1^{k_1} \cdots X_j^{k_j}$$

we have

$$\lambda(k_1, \cdots, k_{j-2}, k_{j-1}, k_j) = \lambda(k_1, \cdots, k_{j-2}, g-k_{j-1}, g-k_j)$$

and similarly for any other pair of the $k_i$. We can see at once by putting $X_1 = X_2 = \cdots = X_j = 0$, that $\lambda(0, 0, \cdots, 0) = 1$. Hence $\lambda(g, g, 0, 0, \cdots, 0) = 1$ and so on. It follows that, for $j \geq 2$, $P_j(n)$ is of degree exactly $g$ in every $X_i$.

Next, we see that, in the sum on the right-hand side of (6), the factor $(1-X_j)^n$ occurs in the denominator only in the term in which $m=1$, $h_1 = n$, i.e. the term corresponding to the partition of $n$ into $n$ units. But the factor $(1-X_j)^n$ occurs in $\beta_j(1)\beta_j(2) \cdots \beta_j(n)$ and so

$$\beta_j(1) \cdots \beta_j(n)$$

and

$$\sum_{k_j=0}^g \lambda(k_1, \cdots, k_j)$$

is the coefficient of $\prod_{i=1}^{j-1} X_i^{k_i}$ in the double product on the right-hand side of (12). Also, putting

$$X_1 = X_2 = \cdots = X_{j-1} = 1,$$

we have

$$P_j(1, 1, \cdots, 1; n) = \sum_{k_1, \cdots, k_j=0}^g \lambda(k_1, \cdots, k_j) = (n!)^{j-1}.$$
Again
\[ P_i(X_1, \cdots, X_{i-1}, 0; n) = P_{i-1}(X_1, \cdots, X_{i-1}; n) \]
and so
\[ \lambda(k_1, \cdots, k_{i-1}, 0) = \lambda(k_1, \cdots, k_{i-1}). \]
By (10), \( \lambda(k_1) = 0 \) unless \( k_1 = 0 \). Hence
\[ \lambda(k_1, 0, 0, \cdots, 0) = 0, \]
unless \( k_1 = 0 \). Thus there is no term in \( P_j \) which consists of a power of one \( X \) only, i.e. apart from the term of zero degree, viz. 1, every term contains at least two of the \( X \). A number of other properties of the \( \lambda \) may be obtained similarly.

From (8) and (9), it follows that
\[
(13) \quad nP_j(n) = \sum_{m=1}^{n} \frac{\beta_j(n - m + 1) \cdots \beta_j(n)}{\beta_j(m)} P_j(n - m).
\]
For \( m \geq 2 \), the factor \( 1 - X_i \) occurs at least once more in the numerator of
\[ \frac{\beta_j(n - m + 1) \cdots \beta_j(n)}{\beta_j(m)} \]
than in the denominator. Hence
\[ nP_j(n) = \frac{\beta_j(n)}{\beta_j(1)} P_j(n - 1) + \beta_j(1)T, \]
where \( T \) is a polynomial in the \( X_i \).

For a small value of \( m \), we can find the terms containing \( X_i^m \) in \( P_j(n) \) as follows. It is easily verified that
\[ G_j(X_iY)G_{j-1}(Y) = G_j(Y) \]
and so
\[ \sum_{n=0}^{\infty} Q_j(n)Y^n = \left\{ \sum_{l=0}^{\infty} Q_j(l)X_i^lY^l \right\} \left\{ \sum_{s=0}^{\infty} Q_{j-1}(s)Y^s \right\}, \]
whence
\[ (1 - X_i^n)Q_j(n) = \sum_{l=0}^{n-1} X_i^lQ_j(l)Q_{j-1}(n - l), \]
that is
\[(14) \quad P_j(n) = \sum_{l=0}^{n-1} X_i^l \left\{ \prod_{m = l+1}^{n-1} \left(1 - X_i^m \right) \right\} \frac{\beta_{j-1}(n - l + 1) \cdots \beta_{j-1}(n)}{\beta_{j-1}(1) \cdots \beta_{j-1}(l)} P_j(l) P_{j-1}(n - l), \]

where, as usual, each empty product denotes unity. The terms in \(X_i^n\) occur in the first \(m+1\) terms on the right and can be expressed in terms of \(P_j(l)\) and \(P_{j-1}(n-l)\) for \(1 \leq l \leq m\). Thus the term in \(X_i\) is

\[X_i \left\{ \frac{\beta_{j-1}(n)}{\beta_{j-1}(1)} \right\} P_{j-1}(n - 1) - P_{j-1}(n) \} \cdot \]

4. Calculation of \(P_j(2)\) and \(P_j(3)\). By (6) and (9),

\[P_j(2) = \frac{\beta_j(1)\beta_j(2)}{2} \left\{ \frac{1}{\{\beta_j(1)\}^2} + \frac{1}{\beta_j(2)} \right\} \]

\[= \frac{1}{2} \left( \frac{\beta_j(2)}{\beta_j(1)} + \beta_j(1) \right) \]

\[= \frac{1}{2} \left\{ \prod_{i=1}^j (1 + X_i) + \prod_{i=1}^j (1 - X_i) \right\} \]

\[= 1 + \sum X_1 X_2 + \sum X_1 X_2 X_3 X_4 + \cdots. \]

Similarly, since \(3 = 2 + 1 = 1 + 1 + 1\), we have

\[P_j(3) = \beta_j(1)\beta_j(2)\beta_j(3) \left\{ \frac{1}{6\{\beta_j(1)\}^2} + \frac{1}{2\beta_j(1)\beta_j(2)} + \frac{1}{3\beta_j(3)} \right\} \]

\[= \frac{1}{6} \left\{ \beta_j(2)\beta_j(3) \right\} \left\{ \frac{1}{\{\beta_j(1)\}^2} + 3\beta_j(3) + 2\beta_j(1)\beta_j(2) \right\} \]

\[= \frac{1}{6} \left\{ \prod_{i=1}^j (1 + X_i)(1 + X_i + X_i^2) + 3 \prod_{i=1}^j (1 - X_i) \right. \]

\[+ 2 \prod (1 - X_i)(1 - X_i) \left\} . \right\}

Now

\[\prod (1 + 2X_i + 2X_i^2 + X_i^3) + 3 \prod (1 - X_i^3) + 2 \prod (1 - X_i - X_i^2 + X_i^3) \]

\[= 3 \prod (1 + X_i) + 3 \prod (1 - X_i^3) + \sum_{a=2}^{j} \{2^a + (-1)^a2 \}
\]

\[\cdot \sum X_1 \cdots X_a(1 + X_1) \cdots (1 + X_a)(1 + X_{a+1}) \cdots (1 + X_j) \]

and so
Macmahon (loc. cit.) gives the above form of \( P_2(2) \), but dismisses \( P_2(3) \) with the remark that it is very complex.

From the above, we have

\[
P_2(2) = 1 + X_1X_2, \quad P_2(2) = 1 + X_1X_2 + X_2X_3 + X_3X_1
\]

and

\[
P_3(3) = 1 + X_1^2X_2 + X_1X_2(1 + X_1)(1 + X_2),
\]

\[
P_3(3) = 1 + X_1^2X_2 + X_2^2X_3 + X_3^2X_4
\]

\[
+ X_1X_2X_3(1 + X_1)(1 + X_2)(1 + X_3) \left\{ \sum X_1 - 2 + \sum \frac{1}{X_1} \right\}.
\]

The formulae (6) and (9) enable one to evaluate \( P_j(n) \) for small \( j \) and \( n \) and, in particular, to pick out the coefficient of any given term.

5. The case \( j = 2 \). By (12), we see that

\[
P_2(X_1, 1; n) = \prod_{m=2}^{n} (1 + X_1 + X_1^2 + \cdots + X_1^{n-1}) = \prod_{m=2}^{n} \left( \frac{1 - X_1^n}{1 - X_1} \right).
\]

We see then that

\[
P_2(X_1, X_2; n) = \prod_{m=2}^{n} \left( \frac{1 - X_1^mX_2^m}{1 - X_1X_2} \right)
\]

vanishes when \( X_2 = 1 \) and similarly when \( X_1 = 1 \). It also vanishes in virtue of (10), when \( X_1 = 0 \) and when \( X_2 = 0 \). It follows that

\[
P_2(X_1, X_2; n) = \prod_{m=2}^{n} \left( \frac{1 - X_1^mX_2^m}{1 - X_1X_2} \right)
\]

\[
+ X_1X_2(1 - X_1)(1 - X_2)M(X_1, X_2; n),
\]

where \( M \) is a polynomial in \( X_1 \) and \( X_2 \). Since

\[
X_1^2X_2^2P_2(X_1^{-1}, X_2^{-1}; n) = P_2(X_1, X_2; n)
\]
and a similar relation is true for the first term on the right-hand side of (15), we must have
\[ X_1^{a-3} X_2^{a-3} M(X_1^{-1}, X_2^{-1}; n) = M(X_1, X_2; n) \]
and so \( M \) is of degree at most \( g - 3 \) in \( X_1 \) and \( X_2 \).

For a fixed \( j \), the recurrence formula (13) provides a slightly less laborious means of finding \( P_j(n) \) than does (6). If we write \( Z = X_1 X_2 \), and
\[ \zeta_\mu = 1 + Z + Z^2 + \cdots + Z^\mu, \]
the values of \( P_2(4) \) and \( P_2(5) \) found from (13) are
\[ P_2(4) = \zeta_3 \zeta_4 - Z \zeta_5 \beta_2(1) - Z \zeta_5 \beta_2(2) \]
\[ = (1 + Z^2) \zeta_4 + Z \zeta_5 (X_1 + X_2) + Z \zeta_5 (X_1^2 + X_2^2) \]
and
\[ P_2(5) = \zeta_3 \zeta_4 \zeta_5 - Z \zeta_6 \beta_2(1) - Z \zeta_6 \beta_2(2) \]
\[ - Z (1 + Z^2) \zeta_6 \beta_2(3) - Z \zeta_5 \beta_2(4) \]
\[ = 1 + Z + 2Z^2 + 3Z^3 + 4Z^4 + 6Z^5 + 4Z^6 \]
\[ + 3Z^7 + 2Z^8 + Z^9 + Z^{10} \]
\[ + Z \zeta_5 \beta_4 (X_1 + X_2) + Z \zeta_5 \beta_4 (X_1^2 + X_2^2) \]
\[ + Z (1 + Z^2) \zeta_6 (X_1^3 + X_2^3) + Z \zeta_5 \beta_4 (X_1^4 + X_2^4). \]
The detailed calculations have no point of interest.

6. Consequences in partition-theory. If
\[ \frac{1}{(1 - X)(1 - X^2) \cdots (1 - X^n)} = 1 + \sum_{t=1}^{\infty} p_n(t)x^t, \]
then \( p_n(t) \) is the number of partitions of \( t \) into parts not greater than \( n \). It is well known (see, for example, Hardy and Wright, Theory of numbers, 3d ed., Oxford, 1955, Theorem 343) that \( p_n(t) \) is also the number of partitions of \( t \) into not more than \( n \) parts. From the definition of \( Q_j(n) \) and \( P_j(n) \), we see that
\[ q(n_1, \cdots, n_j; n) = \sum_{k_1, \cdots, k_j=0}^{g} \lambda(k_1, \cdots, k_j) \prod_{i=1}^{j} p_n(n_i - k_i). \]
Hence, if we calculate \( P_j(n) \), we can express \( q(n_1, \cdots, n_j; n) \) in terms of the \( p_n \). Again
\[ r(n_1, \cdots, n_j; n) = \sum_{k_1, \cdots, k_j=0}^{g} \lambda(k_1, k_2, \cdots, k_{j-1}, g - k_j) \prod_{i=1}^{j} p_n(n_i - k_i). \]
7. An asymptotic expansion for large \( n \). For fixed \( X_i \) such that \( |X_i| < 1 \) (\( 1 \leq i \leq j \)) we can find an asymptotic expansion of \( Q_j(n) \) for large \( n \). For simplicity, we confine ourselves to the case in which \( j = 2, X_1 \) and \( X_2 \) are real and positive and the ratio of their logarithms is not rational, so that \( X_1^v = X_2^u \) is impossible for any positive integral \( u \) and \( v \). In the complex \( Y \)-plane, \( G_2(Y) \) has a simple pole at each of the points

\[
X_1^{-t_1}X_2^{-t_2}\quad (t_1, t_2 \geq 0).
\]

If we write \( \delta = \min(|X_1|^{-1}, |X_2|^{-1}) \),

\[
\phi(\alpha, X) = \prod_{k=0}^{\infty} (1 - \alpha X^k)^{-1},
\]

\[
J = \phi(X_1, X_1)\phi(X_2, X_2) \prod_{k_1=1}^{\infty} \prod_{k_2=1}^{\infty} (1 - X_1^{k_1}X_2^{k_2})^{-1},
\]

and

\[
K(t_1, t_2; X_1, X_2) = \prod_{k_1=1}^{t_1} \prod_{k_2=1}^{t_2} (1 - X_1^{-k_1}X_2^{-k_2})^{-1} \prod_{k_1=1}^{t_1} \phi(X_1^{-k_1}, X_1) \prod_{k_1=1}^{t_1} \phi(X_1^{-k_1}, X_2),
\]

we find that

\[
G_2(Y) = J \sum_{h=0}^{m+1} \sum_{k_1=0}^{h} \frac{K(k_1, h - k_1; X_1, X_2)}{1 - X_1^{k_1}X_2^{h-k_1}Y}
\]

is regular on and within the circle \( |Y| = \delta^{m+1} \). It follows that

\[
Q_2(n) = J \sum_{h=0}^{m} \sum_{k_1=0}^{h} K(k_1, h - k_1; X_1, X_2)n^{k_1}X_1^{n(k-k_1)}X_2^{n(k-k_1)} + O(n^{m+1}),
\]

where the \( O(\quad) \) symbol refers to the passage of \( n \) to infinity.

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