A NOTE ON THE ALGEBRA OF BOUNDED FUNCTIONS. II
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1. Let $K$ be a commutative $B^*$-algebra with identity 1 (with $\|k^*k\| = \|k\|^2$ for all $k \in K$, and $\|1\| = 1$). Then $K$ is equivalent (isomorphic in a norm and * preserving manner) to the algebra $C(M)$ of all continuous complex-valued functions on the compact Hausdorff space $M$ (its structure space) [1; 2]. In [5] we have given necessary and sufficient conditions that $K$ be equivalent to $B(X)$, the ring of all bounded complex-valued functions on the discrete space $X$. These conditions were ideal-theoretic, involving the annulets (annihilating ideals) of $K$, and did not depend on the representation $C(M)$. Using the representation $C(M)$ two further characterizations of $B(X)$ are given in [3], one involving the properties of the space $M$, and the other the notion of projection. The characterizations in [3] are derived independently of the one in [5], and in fact no attempt is made in [3] to relate directly the ideal-theoretic conditions with the notions used there. In this note, we show how the characterizations in [3] can be derived from the characterization in [5] by relating directly the ideal-theoretic properties of $K$ with the properties of the structure space $M$, and with the idea of projection. In particular, the lattice of annulets of $K$ is anti-isomorphic to the lattice of regular open sets in $M$ (Lemma 1). Another characterization of $B(X)$ (Theorem 4) is a byproduct of our procedure.

2. The notation will follow that in [5]. If $G \leq K$ then $R(G)$ is the set of all functions $k \in K = C(M)$ such that $kg = 0$ for all $g \in G$. Such ideals were called annulets. $N(G)$ is the set of $y \in M$ such that $g(y) = 0$ for all $g \in G$. If $S \subseteq M$, then $A(S)$ is the set of functions $f$ such that $f(x) = 0$ for all $x \in S$. Since $N(G) = \bigcap_{g \in G} N(g)$, it is a closed set. Now by following the arguments of Lemma 1 of [5], and using the fact that if $O$ is open in a compact space and $x \in O$, there exists a function $f \in K$ with $f(x) = 1$ and $f(O^c) = 0$, we have

\begin{enumerate}
  \item $R(G) = A \left[ N(G)^c \right]$ for $G \leq K$;
  \item $R[A(S)] = A(S')$ if $S$ is a closed subset of $M$;
  \item $N[A(S)] = S$ if $S$ is a closed subset of $M$.
\end{enumerate}

Now (1) and (2) show that an annulet of $K$ is the set of all functions vanishing on an open set of $M$ and conversely. Since $A(S) = A(S)$

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852
and $S \leq \text{int } \overline{S} \leq \overline{S}$ for any open set $S$, an annulet is the set of functions vanishing on a regular open set and conversely. $(S)$ is a regular open set if $S = \text{int } \overline{S}$ where $\text{int } \overline{S}$ is the largest open set in $\overline{S}$. If $S$ and $T$ are regular open sets such that $A(S) = A(T)$, then $\overline{S} = \overline{T}$ by (3) and hence $S = \text{int } \overline{(S)} = \text{int } \overline{(T)} = T$. We have:

**Lemma 1.** The lattice of annulets of $K$ is anti-isomorphic to the lattice of regular open sets in $M$.

**Lemma 2.** The following are equivalent:

1. The sum of two annulets is an annulet.
2. Every annulet is generated by an idempotent.
3. Every regular open set in $M$ is closed.

**Proof.** Assume (1) and let $A(S)$ be any annulet, where $S$ is a regular open set. Let $T = S'$. Then $T$ is a regular open set such that $S \cap T = 0$. Now $A(S) \cup A(T)$ (the smallest annulet containing $A(S)$ and $A(T)$) = $A(S) + A(T)$ since this is an annulet by (1). But $A(S) \cup A(T) = A(S \cap T) = A(0) = K$ by Lemma 1. Since $A(S) \cap A(T) = 0$ we have $A(S) \oplus A(T) = K$, and it follows as in Theorem 3 of [5] that $A(S) = eK$ where $e$ is the characteristic function of $S'$.

Now assume (2), let $S$ be a regular open set, and $A(S)$ the corresponding annulet. Then $A(S) = eK$, so $N(e) = N(eK) = N[A(S)] = \overline{S}$. Hence $S$ is open and closed. Then $S = \text{int } \overline{S} = \overline{S}$, and $S$ is closed.

Now assume (3) and let $S, T$ be regular open sets ($S \cap T$ is also regular). Since $S$ is open and closed, $A(S)$ is generated by the characteristic function of $S'$. Then $A(T), A(S \cap T)$ are similarly generated by idempotents, and the proof can now be completed as in the proof of Theorem 2 of [5] to show $A(S) + A(T)$ is the annulet $A(S \cap T)$.

**Lemma 3.** The following statements are equivalent:

1. Every nonzero closed ideal of $K$ contains a minimal ideal.
2. For each $k \neq 0$, there exists a minimal projection $e$ such that $ke \neq 0$.
3. The space $M$ contains a dense subset of isolated points.

**Proof.** Assume (1) and let $k \in K, k \neq 0$. Let $J = Kk > 0$. Then $J$ contains a minimal ideal of form $Ke$ for a projection $e$ [4, p. 64] since any idempotent in $K = C(M)$ is obviously self-adjoint. Now $Ke$ is a simple commutative ring with identity, hence a field. By the Gelfand-Mazur theorem $Ke$ is isomorphic to the complex numbers, so that $e$ is a minimal projection. If $ke = 0$, $(Kk)e = 0$, $Je = 0$, $e^2 = e = 0$, a contradiction, so that (2) follows.

The fact that (2) implies (3) is proven in Theorem 2 of [3].

Now assume (3), and let $J$ be a closed ideal $\neq 0$ so that $k \neq 0, k \in J$. If $ke \neq 0$ for some minimal projection $e$, then $ke = \lambda e$ for com-
plex $\lambda$, $e=\lambda^{-1}ke\subseteq J$ and $J$ contains the ideal $Ke$ which is obviously minimal, since it is a field. Assume if possible that $ke=0$ for all minimal projections $e$. Let $X_0$ be the dense set of isolated points. Each minimal projection $e$ is the characteristic function of a point of $X_0$ [3, Theorem 2]. Hence $k(x_0)=0$ for all $x_0\in X_0$, so that $k=0$, since $X_0$ is dense in $M$. This contradicts $k\neq 0$ in $K$, and completes the proof.

Lemmas 2 and 3 and the characterization of [5] give us:

**Theorem 4.** $K$ is equivalent to $B(X)$ if and only if

1. The structure space $M$ contains a dense set $X_0$ of isolated points.
2. Every regular open set in $M$ is closed.

3. In [3] it was shown that $K=C(M)$ is equivalent to $B(X)$ if and only if either of the following sets of conditions is satisfied:

A. (1) $M$ contains a dense subset $X_0$ of isolated points.
(2) If $Q\subseteq X_0$, then there exists an open and closed subset $S$ of $M$ such that $Q\subseteq S$ and $S\cap X_0=Q$.

B. (1) For each $k\neq 0$ in $K$ there exists a minimal projection $e$ with $ke\neq 0$.
(2) For each subcollection $A$ of the collection $P$ of minimal projections, there exists a projection $e_A$ such that $e_Ae_p=e_p$ for $p\in A$ and $e_Ae_p=0$ for $p\notin A$.

The conditions of A are equivalent to those of B directly. This is essentially proved in Theorem 2 of [3]. That the (1)’s are equivalent is pointed out in our Lemma 3. The (2)’s, for example, are each obviously equivalent to the statement: If $A$ is a subset of $X_0$, there exists a function $e(x)$ in $C(M)$ such that $e(x)=1$ if $x\in A$, and $e(x)=0$ if $x\in X_0-A$.

Now assume that the conditions of Theorem 4 hold and let $Q\subseteq X_0$. Then $Q$ is open and $Q\subseteq \text{int } Q\subseteq \overline{Q}$. But since each subset of $X_0$ is open (as a union of points which are open) it is also closed in the relative topology of $X_0$. Hence $Q=X_0\cap \overline{Q}$ and this implies $Q=X_0\cap \text{int } \overline{Q}$. Then $S=\text{int } \overline{Q}$ is open and closed (being a regular open set) and the conditions of $A$ are satisfied.

Now assume the conditions of $A$, and let $S$ be a regular open set in $M$. Let $Q$ be the complement of $X_0\cap S$ in $X_0$. Then $(X_0\cap S)\cup \overline{Q} = X_0 = M$. Now by (2) of $A$, there exists an open and closed set $T$ such that $X_0\cap S\subseteq T$, but $Q\subseteq T'$. Thus $\overline{X_0\cap S}\subseteq T$, $\overline{Q}\subseteq T'$, and $(X_0\cap S)\cap \overline{Q} = 0$, and $\overline{X_0\cap S}$ and $\overline{Q}$ are open, as well as closed. Now since $S\cap Q=0$ and $S$ is open, $S\cap \overline{Q}=0$. Since $\overline{Q}$ is open $S\cap \overline{Q}=0$. Hence $S\subseteq T'$ and $S=\overline{X_0\cap S}$. Thus $S=X_0\cap S$ and $S$ is open. Then $S=\text{int } (S) = S$, and $S$ is closed. Hence the conditions of Theorem 4 are equivalent to $A$ and $B$. 


LIE SIMPLICITY OF A SPECIAL CLASS OF ASSOCIATIVE RINGS

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Given an associative ring $A$, by introducing a new multiplication we can form from it a new ring called the Lie ring of $A$. This multiplication is defined by $[a, b] = ab - ba$ for all $a, b \in A$. If $U$ is an additive subgroup of $A$ and if for arbitrary $u \in U$, $a \in A$, $ua - au \in U$, then $U$ is said to be a Lie ideal of $A$. If $X, Y$ are additive subgroups of $A$ then by $[X, Y]$ we mean the additive subgroup generated by all the elements $xy - yx$, where $x \in X$, $y \in Y$. An additive subgroup $U$ of $[A, A]$ is said to be a proper Lie ideal of $[A, A]$ if $U \neq [A, A]$ and if $[U, [A, A]] \subseteq U$.

In [4], Herstein proved that if $A$ is a simple ring of characteristic not 2 or 3, and if $U$ is a proper Lie ideal of $[A, A]$, then $U$ is contained in $Z$, the center of $A$. In this paper we settle the question in the open case where $A$ is a simple ring of characteristic 2 or 3. The above theorem becomes sharpened to:

**Theorem 1.** If $A$ is a simple ring and if $U$ is a proper Lie ideal of $[A, A]$, then $U$ is contained in the center of $A$, except for the case where $A$ is of characteristic 2 and 4 dimensional over its center, a field of characteristic 2.

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1 The results of this paper will comprise the beginning portion of a thesis, which will be presented to the Faculty of the Graduate School of the University of Pennsylvania in partial fulfillment of the requirements for the degree of Doctor of Philosophy.