A NONMODULAR COMPACT CONNECTED TOPOLOGICAL LATTICE[1]

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The purpose of this note is to present an example of a nonmodular topological lattice which is compact and connected. The question of the existence of such a lattice was raised by Professor A. D. Wallace in a communication to me concerning his researches in topological lattices. The example will be a compact, connected portion of three space in its metric topology.

Let $L=\{(x, y, z)\in R^3 \mid 0\leqq x\leqq 1,\ 0\leqq y\leqq 1, \text{ and } 0\leqq z\leqq x(1-y)\}$ and define the relation $\leqq$ on L such that $(x_1, y_1, z_1)\leqq(x_2, y_2, z_2)$ if and only if (1) $x_1\leqq x_2$, (2) $y_1\leqq y_2$, and (3) $z_1+x_1y_1\leqq z_2+x_2y_2$. Clearly $\leqq$ is a partial order on L and L is a compact, connected portion of R^3 in its metric topology.

For (x_1, y_1, z_1) and (x_2, y_2, z_2) define $a_1=x_1\cup x_2$, $b_1=y_1\cup y_2$, and $c_1=[\bigcup_{i=1,2}(z_i+x_iy_i-a_1b_1)]\cup 0$ where $\cup$ is the lattice operation maximum on the real line under the natural order. Clearly $0\leqq a_1\leqq 1$ and $0\leqq b_1\leqq 1$. Since $z_i\leqq x_i(1-y_i)=x_i-x_iy_i$, $z_i+x_iy_i\leqq x_i\leqq a_1$. Thus $z_i+x_iy_i-a_1b_1\leqq a_1-a_1b_1=a_1(1-b_1)$. Clearly $0\leqq a_1(1-b_1)$, thus $0\leqq c_1=[\bigcup_{i=1,2}(z_i+x_iy_i-a_1b_1)]\cup 0\leqq a_1(1-b_1)$ and $(a_1, b_1, c_1)\in L$. Clearly (a_1, b_1, c_1) is an upper bound of (x_1, y_1, z_1) and (x_2, y_2, z_2). If $(u, v, w)\in L$ is another upper bound, then $u\geqq x_i$, $v\geqq y_i$, and $w+uv\geqq z_i+x_iy_i$ for each i. Immediately $u\geqq a_1$, $v\geqq b_1$, $w+uv\geqq 0+a_1b_1$, and $w+uv\geqq(z_i+x_iy_i-a_1b_1)+a_1b_1$. Thus $u\geqq a_1$, $v\geqq b_1$, and $w+uv\geqq c_1+a_1b_1$; and (a_1, b_1, c_1) is the least upper bound of (x_1, y_1, z_1) and (x_2, y_2, z_2).

Similarly for (x_1, y_1, z_1) and (x_2, y_2, z_2) define $a_2=x_1\cap x_2$, $b_2=y_1\cap y_2$ and $c_2=[\bigcap_{i=1,2}(z_i+x_iy_i-a_2b_2)]\cap[a_2(1-b_2)]$ where $\cap$ is the lattice operation minimum on the real line under the natural order. Clearly $0\leqq a_2\leqq 1$ and $0\leqq b_2\leqq 1$. Since $z_i\geqq 0$, $x_i\geqq a_2\geqq 0$, and $y_i\geqq b_2\geqq 0$; then $z+x_iy_i-a_2b_2\geqq 0$. Clearly $a_2(1-b_2)\geqq 0$, thus

$$a_2(1-b_2)\geqq c_2=\Big[\bigcap_{i=i,2}(z_i+x_iy_i-a_2b_2)\Big]\cap[a_2(1-b_2)]\geqq 0$$

and $(a_2, b_2, c_2)\in L$. Clearly (a_2, b_2, c_2) is a lower bound of (x_1, y_1, z_1) and (x_2, y_2, z_2). If $(u, v, w)\in L$ is another lower bound, then $u\leqq x_i$, $v\leqq y_i$, and $w+uv\leqq z_i+x_iy_i$ for each i. Immediately $u\leqq a_2$, $v\leqq b_2$,

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$w+uv \leqq u(1-v)+uv=u \leqq a_2=a_2(1-b_2)+a_2b_2$, and $w+uv \leqq (z_i+x_iy_i-a_2b_2)+a_2b_2$. Thus $u \leqq a_2$, $v \leqq b_2$, and $w+uv \leqq c_2+a_2b_2$, and (a_2, b_2, c_2) is the greatest lower bound of (x_1, y_1, z_1) and (x_2, y_2, z_2).

Hence, under the partial order $\leqq$, L is a lattice with $(x_1, y_1, z_1) \cup (x_2, y_2, z_2)=(a_1, b_1, c_1)$ and $(x_1, y_1, z_1) \cap (x_2, y_2, z_2)=(a_2, b_2, c_2)$. It is clear that the functions $\cup$ and $\cap$ from $L \times L$ to L are continuous functions in the metric topology, and that $(L, \leqq)$ is a topological lattice in this topology.

It remains to show that this lattice is not modular. Denote $\alpha=(0, 1, 0)$, $\beta=(1, 0, 0)$, $\gamma=(1, 0, 1)$, $0=(0, 0, 0)$, and $I=(1, 1, 0)$. Then clearly $\beta<\gamma$, and the trivial computations imply that $\alpha \cap \beta=\alpha \cap \gamma=0$ and $\alpha \cup \beta=\alpha \cup \gamma=I$. Thus 0, α, β, γ, I constitute a non-modular five lattice and L is not modular.

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