A NONMODULAR COMPACT CONNECTED TOPOLOGICAL LATTICE

DON E. EDMONDSON

The purpose of this note is to present an example of a nonmodular topological lattice which is compact and connected. The question of the existence of such a lattice was raised by Professor A. D. Wallace in a communication to me concerning his researches in topological lattices. The example will be a compact, connected portion of three space in its metric topology.

Let \( L = \{ (x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, \text{ and } 0 \leq z \leq x(1-y) \} \) and define the relation \( \leq \) on \( L \) such that \( (x_1, y_1, z_1) \leq (x_2, y_2, z_2) \) if and only if (1) \( x_1 \leq x_2 \), (2) \( y_1 \leq y_2 \), and (3) \( z_1 + x_1 y_1 \leq z_2 + x_2 y_2 \). Clearly \( \leq \) is a partial order on \( L \) and \( L \) is a compact, connected portion of \( \mathbb{R}^3 \) in its metric topology.

For \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \) define \( a_1 = x_1 \cup x_2, b_1 = y_1 \cup y_2, \) and \( c_1 = [\bigcup_{i=1,2} (z_i + x_i y_i - a_i b_i)] \cup 0 \) where \( \cup \) is the lattice operation maximum on the real line under the natural order. Clearly \( 0 \leq a_1 \leq 1 \) and \( 0 \leq b_1 \leq 1 \). Since \( z_i \leq x_i(1-y_i) = x_i - x_i y_i, \) \( z_i + x_i y_i \leq x_i \leq a_1 \). Thus \( z_i + x_i y_i - a_1 b_1 \leq a_1 - a_1 b_1 = a_1 (1 - b_1) \). Clearly \( 0 \leq a_1 (1 - b_1) \), thus \( 0 \leq c_1 = [\bigcup_{i=1,2} (z_i + x_i y_i - a_1 b_1)] \cup 0 \leq a_1 (1 - b_1) \) and \( (a_1, b_1, c_1) \in L \). Clearly \( (a_1, b_1, c_1) \) is an upper bound of \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \). If \( (u, v, w) \in L \) is another upper bound, then \( u \geq x_i, v \geq y_i, \) and \( w + uv \geq z_i + x_i y_i \) for each \( i \). Immediately \( u \geq a_1, v \geq b_1, w + uv \geq a_1 b_1, \) and \( w + uv \geq (z_i + x_i y_i - a_1 b_1) + a_1 b_1 \). Thus \( u \geq a_1, v \geq b_1, \) and \( w + uv \geq c_1 + a_1 b_1 \); and \( (a_1, b_1, c_1) \) is the least upper bound of \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \).

Similarly for \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \) define \( a_2 = x_1 \cap x_2, b_2 = y_1 \cap y_2, \) and \( c_2 = [\bigcap_{i=1,2} (z_i + x_i y_i - a_2 b_2)] \cap [a_2 (1 - b_2)] \) where \( \cap \) is the lattice operation minimum on the real line under the natural order. Clearly \( 0 \leq a_2 \leq 1 \) and \( 0 \leq b_2 \leq 1 \). Since \( z_i \geq 0, x_i \geq a_2 \geq 0, \) and \( y_i \geq b_2 \geq 0 \); then \( z + x_i y_i - a_2 b_2 \geq 0 \). Clearly \( a_2 (1 - b_2) \geq 0 \), thus

\[
a_2 (1 - b_2) \geq c_2 = [\bigcap_{i=1,2} (z_i + x_i y_i - a_2 b_2)] \cap [a_2 (1 - b_2)] \geq 0
\]

and \( (a_2, b_2, c_2) \in L \). Clearly \( (a_2, b_2, c_2) \) is a lower bound of \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \). If \( (u, v, w) \in L \) is another lower bound, then \( u \leq x_i, v \leq y_i, \) and \( w + uv \leq z_i + x_i y_i \) for each \( i \). Immediately \( u \leq a_2, v \leq b_2, \)

Received by the editors February 6, 1956.

1 This work was done under Contract N7-onr-434, Task Order III, Navy Department, Office of Naval Research.

1157
$w + uv \leq u(1 - v) + uv = u \leq a_2 = a_2(1 - b_2) + a_2b_2$, and $w + uv \leq (x_1 + x_2y - a_2b_2) + a_2b_2$. Thus $u \leq a_2$, $v \leq b_2$, and $w + uv \leq c_2 + a_2b_2$, and $(a_2, b_2, c_2)$ is the greatest lower bound of $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$.

Hence, under the partial order $\leq$, $L$ is a lattice with $(x_1, y_1, z_1) \cup (x_2, y_2, z_2) = (a_1, b_1, c_1)$ and $(x_1, y_1, z_1) \cap (x_2, y_2, z_2) = (a_2, b_2, c_2)$. It is clear that the functions $\cup$ and $\cap$ from $L \times L$ to $L$ are continuous functions in the metric topology, and that $(L, \leq)$ is a topological lattice in this topology.

It remains to show that this lattice is not modular. Denote $\alpha = (0, 1, 0)$, $\beta = (1, 0, 0)$, $\gamma = (1, 0, 1)$, $0 = (0, 0, 0)$, and $I = (1, 1, 0)$. Then clearly $\beta < \gamma$, and the trivial computations imply that $\alpha \cap \beta = \alpha \cap \gamma = 0$ and $\alpha \cup \beta = \alpha \cup \gamma = I$. Thus $0$, $\alpha$, $\beta$, $\gamma$, $I$ constitute a non-modular five lattice and $L$ is not modular.