ON INVARIANT MEANS OVER COMPACT SEMIGROUPS[1]

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1. **Introduction and preliminaries.** A mean on a topological semigroup Σ is a positive element of norm 1 in $C(\Sigma)^*$, where $C(\Sigma)$ is the space of real-valued bounded continuous functions on Σ. Such a mean is called invariant if it is invariant under right and left translations by elements of Σ. Results of Numakura [3] on the algebraic structure of a compact semigroup are used to establish necessary and sufficient conditions that a compact semigroup possess an invariant mean. It is shown that a certain subset, called the kernel, of a compact semigroup with a right invariant mean is a direct product in all senses. It is then proved that if a right invariant mean on a compact semigroup is unique, then it is a two-sided invariant mean.

A *semigroup* Σ is a set in which an associative multiplication is defined. A subset Σ' of Σ is a *subsemigroup* if $\Sigma' \cdot \Sigma' \subseteq \Sigma'$. A nonempty subset L of Σ is a *left ideal* if $\Sigma \cdot L \subseteq L$. A *right ideal* R is a nonempty subset of Σ such that $R \cdot \Sigma \subseteq R$. An *ideal* is a set which is both a right and a left ideal. An element $e \in \Sigma$ is an *idempotent* if $e^2 = e$.

If Σ is both a semigroup and a Hausdorff space, and if the mapping $(s, t) \rightarrow st$ from $\Sigma \times \Sigma$ into Σ is continuous in the given topology, then Σ is called a *topological semigroup*. If Σ is a topological semigroup, let $C(\Sigma)$ denote the Banach space of all real-valued, bounded, continuous functions on Σ, with the usual definitions of addition, scalar multiplication, and norm. Let $C(\Sigma)^*$ denote the first conjugate space of $C(\Sigma)$.

An element $M \in C(\Sigma)^*$ is a *mean* for Σ if (i) $M(1) = 1$, where 1 is the identically one function, (ii) $M(x) \geqq 0$ if $x \geqq 0$, where $x \geqq 0$ means $x(s) \geqq 0$ for all $s \in \Sigma$, and (iii) $\|M\| = 1$. (Any two of these conditions imply the third [1].)

If $x \in C(\Sigma)$ and $s \in \Sigma$, define $r_s x$ by $(r_s x)(t) = x(ts)$ for each $t \in \Sigma$. Similarly, define $l_u x$ by $(l_u x)(v) = x(uv)$ for each $v \in \Sigma$. Then the continuity of the semigroup multiplication shows that $r_s x$ and $l_s x$ are in $C(\Sigma)$ for every choice of $x \in C(\Sigma)$ and $s \in \Sigma$.

A mean M for Σ is called *right invariant* if $M(x) = M(r_s x)$ for every $x \in C(\Sigma)$ and $s \in \Sigma$. A mean M is *left invariant* if $M(x) = M(l_s x)$ for

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every $x \in C(\Sigma)$ and $s \in \Sigma$. M is called *invariant* if it is both right and left invariant.

A measure m on Σ will be called *r^*-invariant* when $m(A) = m(As^{-1})$ for all Borel sets A and all $s \in \Sigma$, where $As^{-1} = \{t \in \Sigma : ts \in A\}$.

2. Main result.

THEOREM 1. *The following conditions on a compact semigroup Σ are equivalent.*

A. *There is a right invariant mean M in $C(\Sigma)^*$.*

B. *There is an r^*-invariant measure m on Σ.*

C. *Σ contains exactly one minimal left ideal.*

PROOF. $A \Rightarrow B$. By the representation of linear functionals as integrals, there is a regular Borel measure m on Σ such that $M(x) = \int x(t)dm(t)$ for every $x \in C(\Sigma)$. By the right invariance of M, $\int x(t)dm(t) = \int x(ts)dm(t)$ for each $s \in \Sigma$. Let A be a closed subset of Σ. By the regularity of m, given $\epsilon > 0$, there is an open set U containing A such that $m(U) \leqq m(A) + \epsilon$. Take $x \in C(\Sigma)$ such that $0 \leqq x \leqq 1$, $x(t) = 1$ if $t \in A$, and $x(t) = 0$ if $t \in \Sigma - U$. Then $m(As^{-1}) \leqq \int x(ts)dm(t) = \int x(t)dm(t) \leqq m(U) \leqq m(A) + \epsilon$ for each $s \in \Sigma$. Since ϵ is arbitrary, $m(As^{-1}) = m(A)$ for every closed set A, and therefore for every Borel set.

$B \Rightarrow A$. The integral defined by the measure m will be a right invariant mean.

$B \Rightarrow C$. By Numakura [**3**], a compact semigroup contains a unique minimal two-sided ideal K, called the ***kernel***, which is compact. Since $\Sigma \cdot s \subseteq K$ if $s \in K$, then $m(K) = m\{t : ts \in K\} = m(\Sigma)$. Therefore $m(\Sigma - K) = 0$, where $\Sigma - K$ denotes the complement of K in Σ. Moreover, if L is any minimal left ideal, then $\Sigma \cdot s \subseteq L$ if $s \in L$, so $m(L) = m(\Sigma)$.

Again by [**3**], a compact semigroup contains minimal left (right) ideals, which are compact, K is the set theoretic union of all minimal left (right) ideals of Σ, and any two minimal left (right) ideals either coincide or are disjoint. Therefore, if L and L' are two minimal left ideals of Σ, then $m(L) = m(L') = m(K) = m(\Sigma)$, while $m(K) \geqq m(L) + m(L') = 2m(K)$, since $L \cup L' \subseteq K$. Therefore, Σ can contain only one minimal left ideal.

$C \Rightarrow B$. The unique minimal left ideal of Σ is, of course, its kernel K. Any compact semigroup contains at least one idempotent and K is a compact subsemigroup of Σ, so there is at least one idempotent in K [**3**]. Let e be an arbitrary idempotent in K, then Ke is a minimal left ideal of Σ [**3**]. By the assumed uniqueness, $K = Ke$ for any idempotent $e \in K$. Therefore, given an arbitrary $t \in K$, there is a $u \in K$ such that $t=ue$, so $te=ue^2=ue=t$. That is, any idempotent in K is a right identity on K.

Let R be an arbitrary minimal right ideal of Σ and let e be an arbitrary idempotent in K. Then $R=Re$ is a group [2]. Let A be an index set, and let $E'=\{e_\alpha:\alpha\in A\}$ be the set of idempotents in K. Then for each $\alpha\in A$, $R_\alpha=e_\alpha K$ is both a minimal right ideal of Σ and a compact group, and the R_α's are mutually isomorphic and disjoint [3].

Let m_α denote the unimodular Haar measure on the compact group R_α. Adjust the scales of the m_α's so that $m_\alpha(R_\alpha)=0$ except for the α's in a finite subset of A, and so that $\sum_\alpha m_\alpha(R_\alpha)=1$. Of course, the adjusted m_α's are, properly speaking, no longer Haar measures, but we shall retain the notation nevertheless. If S is a Borel subset of the compact set K, define $m'(S)=\sum_\alpha m_\alpha(S_\alpha)$, where $S_\alpha=S\cap R_\alpha$.

Now let S be an arbitrary Borel subset of Σ and define $m(S)=m'(S\cap K)$. Take an arbitrary but fixed $s\in\Sigma$ and let $T=\{t:ts\in S\}$. We must show that $m(S)=m(T)$. Because of the definitions of m and m', it is enough to show that $m_\alpha(S_\alpha)=m_\alpha(T_\alpha)$ for each $\alpha\in A$. By definition, $T_\alpha=\{t\in R_\alpha:ts\in S\}$. If $t\in R_\alpha$, then $tv\in R_\alpha$ for all $v\in\Sigma$, so $T_\alpha=\{t\in R_\alpha:ts\in S_\alpha\}$. $ts=te_\alpha s=tu$, where $u=e_\alpha s\in R_\alpha$. Since R_α is a group with identity element e_α, u has an inverse $v\in R_\alpha$ with respect to e_α. Therefore $ts\in S_\alpha\Leftrightarrow tu\in S_\alpha\Rightarrow t\in S_\alpha v\Rightarrow tu\in S_\alpha$. That is, $T_\alpha=S_\alpha v$, so $m_\alpha(T_\alpha)=m_\alpha(S_\alpha v)=m_\alpha(S_\alpha)$ since m_α is right invariant on R_α.

COROLLARY 1. *The following conditions on a compact semigroup* Σ *are equivalent.*

A. *There is an invariant mean in* $C(\Sigma)^*$.

B. *The kernel* K *of* Σ *is a group.*

PROOF. If there is a two-sided invariant mean, then Theorem 1 and its dual result (with left and right interchanged) imply that the kernel K is both the unique minimal left ideal and the unique minimal right ideal. Therefore, there is a single idempotent in K, so K is itself a group, rather than the union of groups.

On the other hand, if K is a group, let m be the unimodular Haar measure on K, and define $m'(S)=m(S\cap K)$ for every Borel subset S of Σ. Then m' has property B of Theorem 1 and the dual left property, so the integral defined by this measure will be two-sided invariant.

Notice further that, by its construction, a two-sided invariant mean on a compact semigroup is unique.

3. Measures on compact semigroups. Notice that the measure used in the course of the proof of Theorem 1 is not necessarily right invariant in the usual sense that $m(A)=m(As)$ for all Borel sets A

and all $s\in\Sigma$. Some results concerning a right invariant measure are possible.

THEOREM 2. *If a compact semigroup Σ has a right invariant measure m, then Σ contains exactly one minimal left ideal, its kernel K, and $m(\Sigma-K)=0$.*

PROOF. We are assuming that $m(A)=m(As)$ for all Borel sets A and all $s\in\Sigma$. If $s\in K$, then $\Sigma\cdot s\subseteq K$, so $m(\Sigma)\leqq m(K)\leqq m(\Sigma)$. Therefore $m(K)\geqq 0$, and $m(\Sigma-K)=0$. Since we have used only the fact that K is a left ideal, we have that $m(\Sigma)=m(K)=m(L)$ for any left ideal L. Hence, as in the proof of Theorem 1, there is only one minimal left ideal.

The converse of Theorem 2 is not necessarily true. In fact, there is a negative answer to the following, posed in a communication from A. D. Wallace: If Σ is a compact semigroup with identity e, is there a nontrivial measure m on Σ and a real-valued function F on Σ such that $m(As)=F(s)\cdot m(A)$ for every Borel set A and every $s\in\Sigma$? Consider the three element semigroup, $\Sigma=\{a, b, e\}$ where $ab=aa=ae=ea=a$, $ba=bb=be=eb=b$, and $ee=e$. Then $K=\{a, b\}$ is the unique minimal left ideal, $\{a\}$ and $\{b\}$ are minimal right ideals, and $\Sigma-K=\{e\}$. Assume $m(\{a\})\neq 0$. Then, for example, $m(\{a\})=F(a)\cdot m(\{aa\})=F(a)\cdot m(\{a\})$, so $F(a)=1$. Similarly, $F(b)=F(e)=1$, so m would have to be strictly right invariant. Then, as in Theorem 2, $m(\Sigma-K)=m(\{e\})=0$. Therefore, $m(\{a\})=m(\{ea\})=m(\{e\})=0$.

A very weak partial converse of Theorem 2 is possible.

THEOREM 3. *If a compact semigroup Σ contains exactly one minimal left ideal, its kernel K, and if $\Sigma=K$, then there is a right invariant measure on Σ.*

PROOF. The measure m' defined in the proof of $C\Rightarrow B$ of Theorem 1 is right invariant under the assumption that $\Sigma=K$.

4. **The kernel.** Since any right invariant mean on a compact semigroup Σ induces a measure on Σ which vanishes on the complement of the kernel K of Σ, the kernel K deserves a more careful scrutiny. We already know that the kernel K of a compact semigroup Σ with right invariant mean is the (set theoretic) union of disjoint, mutually isomorphic, compact groups, R_α, which are also minimal right ideals of Σ. Let E' be the set of idempotents in K. Using E', one can decompose K as a direct product.

THEOREM 4. *Let K be the kernel of a compact semigroup Σ which has* *a right invariant mean. Then K is a direct product in the sense of its algebraic structure, its topology, and its measure.*

PROOF. First notice that E' is a compact subsemigroup of K with the multiplication rule $ab=a$ for all $a, b\in E'$. To prove compactness, take $e_\alpha\in E'$ such that $e_\alpha\to s\in K$. Then, by continuity of the multiplication in Σ, $e_\alpha^2=e_\alpha\to s^2$, so $s^2=s$. That is, $s\in E'$, so E' is closed and hence compact.

Let R be an arbitrary but fixed R_α. Define $K'=E'\times R$, with $(e_\alpha, a)\cdot(e_{\alpha'}, b)=(e_\alpha e_{\alpha'}, ab)=(e_\alpha, ab)$. K' is then clearly a topological semigroup, and is compact as both E' and R are compact.

Define the mapping f from K' to K by $f(e_\alpha, a)=e_\alpha a$. This is certainly a well defined mapping. To show that f is an algebraic homomorphism of K' into K is a straightforward verification. To show that f is an isomorphism onto, take $s\in K$, then there is an α such that $s\in R_\alpha$. Let e be the identity of R, then since R is a right ideal, $es\in R$. Therefore, $f(e_\alpha, es)=e_\alpha es=e_\alpha s=s$, so f is onto. If $f(e_\alpha, a)=f(e_{\alpha'}, b)$, then $e_\alpha a=e_{\alpha'}b\in R_\alpha\cap R_{\alpha'}$, so $e_\alpha=e_{\alpha'}$, and therefore $a=b$, so f is an isomorphism. Since f is $1:1$ and onto, and since f is continuous by the continuity of multiplication, f is a homeomorphism.

All that remains to be shown is that any r^*-invariant measure on K' is a product measure. Let m be such a measure. Define m_1 on E' by $m_1(A)=m(A\times R)$, then m_1 is a regular Borel measure. Define m_2 on R by $m_2(B)=m(E'\times B)$, then m_2 is also a regular Borel measure.

By the r^*-invariance of m, $m(A\times B)=m\{t:ts\in A\times B\}$ for all $s\in K'$. Let $t=(e_\alpha, a)$ and let $s=(e_{\alpha'}, b)$, then $ts=(e_\alpha, ab)$, so $ts\in A\times B$ if and only if both $e_\alpha\in A$ and $ab\in B$, that is, if and only if both $e_\alpha\in A$ and $a\in Bb^{-1}$, where the inverse is with respect to e, the idempotent in R. Hence, $m(A\times B)=m(A\times Bb)$ for every $b\in R$. That is, the r^*-invariance of m implies right invariance of m_2, so m_2 is a multiple of Haar measure on R. Since $m_2(R)=m(E'\times R)=1$, m_2 is actually the Haar measure on R.

For the remainder of the proof, let $\tilde{S}$ denote the complement of S in its containing set. That is, if A is a subset of E', let $\tilde{A}=E'-A$, while if B is a subset of R, let $\tilde{B}=R-B$.

Now let S be an arbitrary Borel subset of E', and define m_S on R by $m_S(B)=m(S\times B)$. For each $b\in R$, $m_S(Bb)=m(S\times Bb)=m(S\times B)=m_S(B)$. That is, m_S is right invariant on R, so it is a multiple of the Haar measure m_2, where $m_S(R)=m(S\times R)$ is the factor.

Therefore, $m(A\times B)=m_A(B)=m(A\times R)\cdot m_2(B)=m(A\times R)\cdot m(E'\times B)$. In a similar fashion it can be shown that $m(A\times\tilde{B})=m(A\times R)\cdot m(E'\times\tilde{B})$, $m(\tilde{A}\times B)=m(\tilde{A}\times R)\cdot m(E'\times B)$, and $m(\tilde{A}\times\tilde{B})=m(\tilde{A}$

$\times R)\cdot m(E'\times \tilde{B})$. Therefore, $m(A\times B)\cdot m(\tilde{A}\times \tilde{B})=m(\tilde{A}\times B)\cdot m(A\times \tilde{B})$.

Now define the product measure $m'(A\times B)=m_1(A)\cdot m_2(B)$. Then

$$
\begin{aligned}
m'(A \times B) \\
&= m_1(A)\cdot m_2(B) = m(A\times R)\cdot m(E'\times B)\\
&= \{m(A\times B)+m(A\times \tilde{B})\}\cdot\{m(A\times B)+m(\tilde{A}\times B)\}\\
&= \{m(A\times B)\}^2+m(A\times \tilde{B})\cdot m(A\times B)+m(A\times B)\cdot m(\tilde{A}\times B)\\
&\qquad + m(A\times \tilde{B})\cdot m(\tilde{A}\times B)\\
&= \{m(A\times B)\}^2+m(A\times \tilde{B})\cdot m(A\times B)+m(A\times B)\cdot m(\tilde{A}\times B)\\
&\qquad + m(A\times B)\cdot m(\tilde{A}\times \tilde{B})\\
&= m(A\times B)\cdot\{m(A\times B)+m(A\times \tilde{B})+m(\tilde{A}\times B)+m(\tilde{A}\times \tilde{B})\}\\
&= m(A\times B)\cdot\{m(A\times R)+m(\tilde{A}\times R)\}\\
&= m(A\times B)\cdot m(E'\times R)\\
&= m(A\times B).
\end{aligned}
$$

Using the ideas developed in the proof of Theorem 4, one can easily construct a semigroup which is equal to its kernel and which has a right invariant mean.

Let E be an arbitrary compact space. Define a multiplication in E by $ab=a$ for all $a, b\in E$. Then, by the obvious associativity and continuity of this product, E is a compact semigroup. Let R be an arbitrary compact group. Form $K=E\times R$, with $(e_\alpha, a)\cdot(e_{\alpha'}, b)=(e_\alpha, ab)$. K is also a compact semigroup. If $R_\alpha=\{(e_\alpha, r): r\in R\}$, then the sets R_α are disjoint compact isomorphic groups, each of which is a minimal right ideal of K, while the only left ideal is K itself. Therefore K is all kernel, and there is a right invariant mean on K.

Let m_1 be an arbitrary regular Borel measure on the compact space E. Let m_2 be the (unique) Haar measure on the compact group R. Define m on K by $m(A\times B)=m_1(A)\cdot m_2(B)$. Then m is easily a regular r^*-invariant Borel measure on K, and so by Theorem 1, m determines a right invariant mean, M_m, on K.

The interesting fact here is the complete lack of uniqueness, for M_m is determined by the measure m, which in turn is determined by the measure m_1 on E. Hence the mean M_m is no more unique than is regular Borel measure on the compact space E, and that is extremely nonunique.

COROLLARY 2. *If a compact semigroup Σ has a unique right invariant mean M, then the kernel K of Σ is a group, and so M is two-sided invariant.*

PROOF. The uniqueness of M implies that E', the set of idempotents in K, consists of a single point. Therefore K is a group.

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LIMITING SETS OF TRAJECTORIES OF A PENDULUM-TYPE SYSTEM

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We consider the system of differential equations

$$\dot{\theta} = z,\ \dot{z} = G(\theta, z); \tag{1}$$

here $G(\theta, z)$ is continuous in (θ, z) and satisfies some condition (such as a Lipschitz condition) which insures the uniqueness of the solution of (1) at each ordinary point. We also suppose that $G(\theta+2\pi, z) = G(\theta, z)$ for all θ, and that $G(\theta, 0) = 0$ has at most a finite number of roots on $0 \leqq \theta < 2\pi$.

It is known (cf. [1, p. 287]) that the solutions of such a system can be studied by means of a cylindrical phase space; i.e., in terms of the curves of the solutions $(\theta(t), z(t))$, the so-called phase trajectories, on the cylinder $r=1$, where (r, θ, z) are cylindrical coordinates. We shall refer to this cylinder $r=1$ as the phase cylinder.

From (1) we obtain the equation

$$z'(\theta) = G(\theta, z)/z(\theta). \tag{2}$$

Suppose $z_1(\theta)$ and $z_2(\theta)$ are solutions of equation (2) such that for $0 \leqq \theta < 2\pi$, $z_1(\theta) > 0$, $z_2(\theta) < 0$, and $z_i(2\pi) < z_i(0)$ for $i = 1, 2$, and consider the region R_{12} of the phase cylinder bounded above by the arc of $z_1(\theta)$ for $0 \leqq \theta < 2\pi$ and the line segment joining $z_1(0)$ and $z_1(2\pi)$, and below by the arc of $z_2(\theta)$ for $0 \leqq \theta < 2\pi$ and the line segment joining $z_2(0)$ and $z_2(2\pi)$. We observe that R_{12} must contain the positive

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