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ON COMMUTATORS AND JACOBI MATRICES[1]

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1. All operators in this paper are bounded linear transformations on a Hilbert space. The commutator C of two operators A and B is defined by

$$C = AB - BA. \tag{1}$$

The closure, W, of the set of values (Cx, x) when $\|x\| = 1$ is a closed convex set (Hausdorff, cf. [8, p. 34]). A complex number z will be said to belong to the interior of W if z is in W and if one of the following conditions holds: (i) If W is two-dimensional, then z does not lie on the boundary of W; (ii) If W is a line segment, then z is not an end point; (iii) W consists of z alone.

It was shown in [4] that if A (or B) is normal, or even semi-normal, so that AA^*-A^*A is semi-definite, then 0 belongs to W, but is not necessarily in the interior of W. (That, for arbitrary A and B, in general 0 need not even belong to W was shown in [2].) In fact, if $A=(a_{ij})$ is defined by $a_{i,i+1}=1$, $a_{ij}=0$ if $j\neq i+1$, then $C=AA^*-A^*A=(c_{ij})$ is the self-adjoint matrix all elements of which are zero except $c_{11}=1$. Consequently, $C\geqq 0$ with a spectrum consisting of $\lambda=0, 1$; hence W is the segment $0\leqq\lambda\leqq 1$ and 0 is not in the interior of W. Moreover, the above C can also be expressed by $C=DA^*-A^*D$ where D is the (self-adjoint) Jacobi matrix $A+A^*$; cf. § 5 below.

The problem to be considered in the present paper is that of determining a sufficient condition guaranteeing that 0 belongs to the interior of W. Such a condition yields information concerning the spec-

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trum of C, at least if C is normal (since W is then the least convex set containing the spectrum of C). Concerning the possible spectra for commutators (not necessarily normal) see [**3**, p. 198].

2. There will be proved the following principal

LEMMA. *Let A be normal with the spectral resolution*

$$A = \int z dK(z). \tag{2}$$

Let S be an arbitrary measurable (here, and in the sequel, "measurable" implies "measurable with respect to $K(z)$") set in the complex plane with a covering $\{\Delta_1, \Delta_2, \Delta_3, \cdots\}$ by pair-wise disjoint measurable sets Δ_n of diameter d_n. Suppose that $H = C + C^*$ is non-negative definite with the self-adjoint square root $H^{1/2}$. Then, if x is any element of Hilbert space, there holds the inequality

$$\left\| H^{1/2} \int_S dKx \right\|^2 \leqq 4\|B\| \|x\|^2 \left(\sum_n d_n \right). \tag{3}$$

Moreover, if A is self-adjoint or unitary, then

$$\left\| H^{1/2} \int_S dKx \right\|^2 \leqq 4\|B\| \|x\|^2 \text{ (meas } S), \tag{4}$$

where the measure refers to the ordinary one-dimensional Lebesgue measure on a segment or a circle according as A is self-adjoint or is unitary.

PROOF OF THE LEMMA. Let Δ denote an arbitrary measurable set in the complex plane and let $\Delta K = \int_\Delta dK$. Multiplication by ΔK on the left and on the right of the equation (1) yields

$$\begin{aligned} \Delta K C \Delta K &= \int_\Delta z dKB\Delta K - \Delta KB \int_\Delta z dK \\ &= \int_\Delta (z - z_0) dKB\Delta K - \Delta KB \int_\Delta (z - z_0) dK, \end{aligned} \tag{5}$$

where z_0 is an arbitrary constant. (Cf. [**1**] and [**5**] for calculations of a somewhat similar nature.) It is clear that

$$\begin{aligned} (\Delta Kx, C\Delta Kx) &= \left(\Delta Kx, \int_\Delta (z - z_0) dKB\Delta Kx \right) \\ &\quad - \left(\Delta Kx, \Delta KB \int_\Delta (z - z_0) dKx \right). \end{aligned} \tag{6}$$

If adjoints are taken in (5), a relation (6′) similar to (6) exists for C^*. Addition of (6) and (6′) followed by a majorization leads to $(\Delta Kx, H\Delta Kx) \leqq 4\|B\|\|\Delta Kx\|^2 d$, where d is the diameter of Δ. Consequently,

$$\|H^{1/2}\Delta Kx\| \leqq 2\|B\|^{1/2}\|\Delta Kx\| d^{1/2}. \tag{7}$$

Next, by (7),

$$\begin{aligned} \left\| H^{1/2} \int_S dKx \right\| &\leqq \sum \left\| H^{1/2} \int_{S\Delta_n} dKx \right\| \\ &\leqq \sum 2\|B\|^{1/2} \left\| \int_{S\Delta_n} dKx \right\| (\text{diam } S\Delta_n)^{1/2}. \end{aligned} \tag{8}$$

In view of the inequalities $\sum_n \|\int_{S\Delta_n} dKx\|^2 = \int_S d\|Kx\|^2 \leqq \|x\|^2$ and diam $S\Delta_n \leqq d_n$, an application of the Schwarz inequality to (8) now yields the desired relation (3). Relation (4) is readily verified to be a consequence of (3) (if one chooses the Δ_n for instance to be rectangles) and the proof of the lemma is now complete.

3. As a consequence of the lemma, there will be derived in this section the following

THEOREM. (i) *Suppose that A is normal with the spectral resolution* (2) *and that*

$$I = \int_S dK, \tag{9}$$

for a measurable set S. If S can be covered by a sequence of pair-wise disjoint measurable sets Δ_n for which $\sum_n$ (diam Δ_n) *can be made arbitarily small, then* 0 *belongs to the interior of the set W associated with C of* (1).

(ii) *Moreover, if A is self-adjoint or unitary, and if there exists a set $S = Z$ of* (*one-dimensional*) *measure zero for which* (9) *holds, then* 0 *belongs to the interior of the set W.*

PROOF OF THE THEOREM. Suppose, if possible, that 0 is not in the interior of W. Then choose θ so that the set $W_\theta (= W \exp(i\theta))$ belonging to $C = A(B\exp(i\theta)) - (B\exp(i\theta))A (= C\exp(i\theta))$ lies entirely in the half-plane $R(z) \geqq 0$. (Cf. [4, p. 129].) It is clear that $H_\theta = C_\theta + C_\theta^* \geqq 0$ and that $\|B\exp(i\theta)\| = \|B\|$. Consequently an application of (3) of the lemma shows that $\|H_\theta^{1/2}x\|^2 \leqq 4\|B\|\,\|x\|^2(\sum_n d_n)$. Hence, in view of the hypothesis, $\|H_\theta^{1/2}x\| = 0$ for every x, that is, $H_\theta^{1/2}$ (hence H_θ) is the zero operator. Therefore

$$C\exp(i\theta) = -(C\exp(i\theta))^*, \tag{10}$$

thus C exp $(i\theta+i\pi/2)$ is self-adjoint. Since $W_{\theta+\pi/2}$ is therefore a segment of the real axis, the set W is a segment containing the origin. If however, 0 is not in the interior of this segment, there exists an angle ϕ for which the set W_ϕ belonging to C exp $(i\phi)$ lies on the real axis with left end point at 0. In view of relation (10), which now holds for $\theta=\phi$, and the fact that C exp $(i\phi)$ is self-adjoint, it follows that C exp $(i\phi)=0$, hence $C=0$, in contradiction to the assumption that 0 is not in the interior of W. This completes the proof of (i) of the theorem.

The proof of part (ii) of the theorem is similar if one notes that (4) now applies to the set $S=Z$ and meas $Z=0$.

4. Various corollaries of the above theorem will be collected in this section.

COROLLARY 1. *If A (or B) is normal with a pure point spectrum then* 0 *is in the interior of the set W.*

The proof is clear from part (i) of the theorem if it is noted that S can be chosen to be a denumerable point set.

COROLLARY 2. *If A (or B) is self-adjoint or unitary with spectrum of (one-dimensional) measure zero then* 0 *is in the interior of W.*

The proof follows from part (ii) of the theorem if it noted that (9) must hold when S is the spectrum of A.

COROLLARY 3. *If $C=AA^*-A^*A$ is semi-definite, then either* (i) $C=0$ *or* (ii) *if $A+A^*$ has the spectral resolution $\int\lambda dE(\lambda)$, then $\int_Z dE<I$ for every (one-dimensional) zero set Z on the real axis.*

It is clear that

$$C = (A + A^*)A^* - A^*(A + A^*), \tag{11}$$

so that $A+A^*$ can be identified with A of the theorem. If now (ii) of Corollary 3 fails to hold, so that $\int_Z dE=I$ for some zero set Z, then part (ii) of the theorem implies that 0 belongs to the interior of W. Since however the end points of W belong to the spectrum of the (semi-definite) operator C, then W must reduce to the single point 0. Therefore $C=0$; that is, (i) of Corollary 3 holds.

5. **On the spectra of Jacobi matrices.** That C of Corollary 3 and (11) need not be zero was shown by the example in §1 where $A+A^*=D=(d_{ij})$ is the Jacobi matrix with $d_{i,i+1}=d_{i+1,i}=1$, $d_{ij}=0$ otherwise. (For a discussion of this type of Jacobi matrix (i.e., one in which the diagonal elements are 0) see [8, p. 236].

More generally if $A=(a_{ij})$ is defined by $a_{i,i+1}=b_i$ and $a_{ij}=0$ for $j\neq i+1$, then $A+A^*=D=(d_{ij})$ is the self-adjoint matrix with $d_{i,i+1}=b_i$, $d_{i+1,i}=b_i^*$, and $d_{ij}=0$ otherwise. Moreover it is easily verified that $C=DA^*-A^*D$ is the diagonal matrix with diagonal elements $|b_1|^2, |b_2|^2-|b_1|^2, |b_3|^2-|b_2|^2, \cdots$. If the b_i satisfy

$$0<|b_1|\leqq|b_2|\leqq|b_3|\leqq\cdots, \tag{12}$$

then D is a Jacobi matrix and C $(\neq 0)$ is non-negative definite. Thus, case (ii) of Corollary 3 applies and if $D=\int\lambda dE(\lambda)$ then $\int_Z dE<I$ for every zero set Z of the λ axis. (In particular, when (12) holds, D cannot have a pure point spectrum, nor can its spectrum be a zero set.)

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