A NOTE ON THE SEPARATION OF CONNECTED SETS BY FINITE SETS

C. E. BURGESS

A connected set $K$ is said to be separated by a subset $H$ of $K$ if $K - H$ is not connected. J. R. Kline has shown that if $n$ is an integer greater than two and the plane continuum $M$ is separated by every subset of $M$ consisting of $n$ points, then $M$ is separated by some set consisting of $n - 1$ points [1, Theorem 5]. A stronger conclusion has been obtained by G. T. Whyburn for the case where $M$ is a locally compact connected metric space. In fact, it follows from Whyburn's results that if every set consisting of $n$ points separates the nondegenerate locally compact connected metric space $M$, then $M$ is a Menger regular curve and contains uncountably many mutually exclusive pairs of points each pair of which separates $M$ [2, p. 313]. It is the purpose of this note to present a proof of a related theorem for a connected topological space.

**Theorem.** If $S$ is a nondegenerate connected topological space$^1$ and $D$ is an open set such that each infinite subset of $D$ contains a finite set that separates $S$, then some pair of points in $D$ separates $S$.

The following two lemmas will be used in the proof of this theorem.

**Lemma 1.** If $S$ is a connected topological space, $M_1$ and $M_2$ are mutually exclusive closed sets such that $M_2$ does not separate $S$, and $H$ is a connected subset of $S - (M_1 + M_2)$ such that some open set contains $M_1$ and lies in $H + M_1$, then $M_1 + M_2$ does not separate $S$.

**Proof.** Suppose $S - (M_1 + M_2)$ is the sum of two mutually separated sets $X$ and $Y$, where $X$ contains the connected set $H$. Since some open set lies in $H + M_1$ and contains $M_1$, it follows that no point of $M_1$ is a limit point of $Y$. This leads to the contradiction that $S - M_2$ is the sum of the two mutually separated sets $X + M_1$ and $Y$.

**Lemma 2.** If $D$ is an open set in a connected topological space $S$, $L$ is a finite subset of $D$ consisting of more than two points such that $S - L$ is the sum of two mutually separated sets $H$ and $K$, and no subset of $D$ with fewer points than $L$ separates $S$, then for each point $p$ of $D - H$ the set $H + L - p$ is connected.

Presented to the Society, September 2, 1955; received by the editors July 22, 1955 and, in revised form, January 16, 1956.

$^1$ The definition of a topological space given in [3] is used here.
Proof. Suppose there is a point $p$ of $D \cdot H$ such that $H + L - p$ is the sum of two mutually separated sets $X$ and $Y$. Since $S - P$ is connected, it follows that both $X$ and $Y$ intersect $L$. Let $n$ denote the number of points in $L$. Since $n > 2$, it follows that one of the sets $X \cdot L + p$ and $Y \cdot L + p$ consists of less than $n$ points. This involves a contradiction since each of these two subsets of $D$ separates $S$.

Proof of theorem. Suppose that no pair of points in $D$ separates $S$. Let $L_1$ be a subset of $D$ such that (1) $S - L_1$ is the sum of two mutually separated sets $H_1$ and $K_1$ and (2) no set in $D$ with fewer points than $L_1$ separates $S$. Let $p_1$ be a point of $K_1 \cdot D$. By Lemma 2, $K_1 + L_1 - p_1$ is connected.

Let $L_2$ be a subset of $D \cdot H_1$ such that (1) $S - L_2$ is the sum of two mutually separated sets $H_2$ and $K_2$, where $K_2$ contains the connected set $K_1 + L_1$, and (2) no set in $D \cdot H_1$ with fewer points than $L_2$ separates $S$. Let $p_2$ be a point of $D \cdot [K_2 - (K_1 + L_1)]$. By Lemma 2, $K_2 + L_2 - p_2$ is connected, and since $K_1 + L_1 - p_1$ is connected and $K_1$ is an open set lying in $K_1 + L_1$, it follows from Lemma 1 that $p_1 + p_2$ does not separate the connected set $K_2 + L_2$.

Let $L_3$ be a subset of $D \cdot H_2$ such that (1) $S - L_3$ is the sum of two mutually separated sets $H_3$ and $K_3$, where $K_3$ contains the connected set $K_2 + L_2$, and (2) no set in $D \cdot H_2$ with fewer points than $L_3$ separates $S$. Let $p_3$ be a point of $D \cdot [K_3 - (K_2 + L_2)]$. By Lemma 2, $K_3 + L_3 - p_3$ is connected, and since $K_2 + L_2 - (p_1 + p_2)$ is connected and $K_2$ is an open set lying in $K_2 + L_2$, it follows from Lemma 1 that $p_1 + p_2 + p_3$ does not separate the connected set $K_3 + L_3$.

By continuing this process indefinitely, a sequence of distinct points $p_1, p_2, p_3, \ldots$ of $D$ can be obtained such that, for each $n$, $p_1 + p_2 + \cdots + p_n$ does not separate the connected set $K_n + L_n$. Since each $H_n + L_n$ is connected, it readily follows that for each $n$, $p_1 + p_2 + \cdots + p_n$ does not separate $S$. This leads to the contradiction that no finite subset of the infinite set $p_1 + p_2 + p_3 + \cdots$ separates $S$.

Corollary. If $n$ is a positive integer and the nondegenerate connected topological space $S$ is separated by every set consisting of $n$ points, then each open set contains a pair of points that separates $S$.

References


University of Utah