PARACOMPACTNESS AND AN EXAMPLE DUE TO F. B. JONES

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At the summer meeting (1955) of the American Mathematical Society, Mary E. Rudin presented an example of a separable normal nonparacompact space. It is the purpose of this note to point out that an example [3] due to F. B. Jones (1937) with an obvious definition of open sets is also such an example. Jones' paper was published before the notion of paracompactness appeared in the literature. The reader is referred to [1; 2] for definitions concerning paracompactness.

EXAMPLE (JONES). Let M_1 denote a subset of the open number interval $I(0, 1)$ of cardinality $\aleph_1$ such that each countable subset of M_1 is an inner limiting set with respect to M_1. Let Z_1 denote the set of all points (x, y) of the number plane such that both x and y are positive rational numbers and $0<x<1$. Furthermore, let α denote a most economical well ordered sequence of the points of M_1, i.e., for p in M_1, p is preceded in α by at most a countable subset of M_1. Let S denote a space whose points are the points of M_1 and Z_1 in which *open sets* are defined as follows:

(1) For p in Z_1, p is an open set.

(2) For p in M_1 such that p has an immediate predecessor in α, an open set containing p is a point set D in M_1+Z_1 such that (a) $D\supset p$ and (b) there exists an interior T of an inverted isosceles triangle with its lower vertex at p and whose base is parallel to the x-axis such that $T\cdot(M_1+Z_1)=D-p$.

(3) For p in M_1 such that p has no immediate predecessor in α, let a denote a point of M_1 such that $a<p$ in α. Now, for a point x of M_1 such that $a<x\leqq p$ in α, let T_x denote the interior of an inverted isosceles triangle with its lower vertex at x and whose base is parallel to the x-axis. An open set D containing p is the set of all points y in S such that either $y=x$ or $y\in T_x\cdot Z_1$.

It is easy to see that S is a separable Hausdorff space. By a slight modification of Jones' argument, it may be shown that S is normal. In his argument where he considers $K\cdot M_1$ to be countable, replace the set D_{1k} by a set Q_{1k} such that $Q_{1k}=Q\cdot M_1$ where Q is an open set in S such that (1) Q is countable, (2) $Q\cdot M_1$ is a closed subset of M_1, and (3) $Q\cdot M_1\cdot H=Q\cdot H=0$.

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It will now be shown that S is not paracompact. Let G denote an open covering of S such that an open set $D \in G$ if and only if either (1) D is a point of Z_1 or (2) there exists p in M_1 such that $D \cdot M_1$ is the set of all points x in M_1 such that $x < p$ in α. Now, suppose that there exists an open refinement G_1 of G which is locally finite. Then at most a finite number of the elements of G_1 intersects M_1. Otherwise, there exists a point p in M_1 and an infinite subcollection G_2 of G_1 such that (1) for g in G_2 and x in M_1 where $p \leqq x$ in α, $g \not\supset x$ and (2) if $x < p$ in α, then there exists g in G_2 such that $g \supset x$; and furthermore, at most a finite number of the elements of G_2 contains x. Thus, for an open set $D \supset p$, D intersects infinitely many of the elements of G_1. This is contrary to the supposition that G_1 is locally finite. If at most a finite number of the elements of G_1 intersects M_1, then some element of G_1 contains uncountably many points of M_1. This is impossible since G_1 is a refinement of G. It follows that S is not paracompact.

Observe that S is not a semi-metric topological space. Thus, the following questions arise naturally.

(1) Is a normal separable semi-metric topological space paracompact?

(2) Is a normal semi-metric topological space paracompact?

If the answer to (1) is "no," then it follows from work of F. B. Jones [3] that the continuum hypothesis is false. An affirmative answer to (1) would free an important result of Jones' from the continuum hypothesis.

Bibliography

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2. C. H. Dowker, *On countably paracompact spaces*, Canadian Journal of Mathematics vol. 3 (1951) pp. 219–224.

3. F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 671–677.

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