PARACOMPACTNESS AND AN EXAMPLE DUE TO F. B. JONES

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At the summer meeting (1955) of the American Mathematical Society, Mary E. Rudin presented an example of a separable normal nonparacompact space. It is the purpose of this note to point out that an example [3] due to F. B. Jones (1937) with an obvious definition of open sets is also such an example. Jones’ paper was published before the notion of paracompactness appeared in the literature. The reader is referred to [1; 2] for definitions concerning paracompactness.

Example (Jones). Let \( M_1 \) denote a subset of the open number interval \( I(0, 1) \) of cardinality \( \mathfrak{K} \) such that each countable subset of \( M_1 \) is an inner limiting set with respect to \( M_1 \). Let \( Z_1 \) denote the set of all points \((x, y)\) of the number plane such that both \( x \) and \( y \) are positive rational numbers and \( 0 < x < 1 \). Furthermore, let \( \alpha \) denote a most economical well ordered sequence of the points of \( M_1 \), i.e., for \( p \) in \( M_1 \), \( p \) is preceded in \( \alpha \) by at most a countable subset of \( M_1 \). Let \( S \) denote a space whose points are the points of \( M_1 \) and \( Z_1 \) in which open sets are defined as follows:

1. For \( p \) in \( Z_1 \), \( p \) is an open set.
2. For \( p \) in \( M_1 \) such that \( p \) has an immediate predecessor in \( \alpha \), an open set containing \( p \) is a point set \( D \) in \( M_1 + Z_1 \) such that (a) \( D \supseteq p \) and (b) there exists an interior \( T \) of an inverted isosceles triangle with its lower vertex at \( p \) and whose base is parallel to the \( x \)-axis such that \( T \cdot (M_1 + Z_1) = D - p \).
3. For \( p \) in \( M_1 \) such that \( p \) has no immediate predecessor in \( \alpha \), let \( a \) denote a point of \( M_1 \) such that \( a < p \) in \( \alpha \). Now, for a point \( x \) of \( M_1 \) such that \( a < x \leq p \) in \( \alpha \), let \( T_x \) denote the interior of an inverted isosceles triangle with its lower vertex at \( x \) and whose base is parallel to the \( x \)-axis. An open set \( D \) containing \( p \) is the set of all points \( y \) in \( S \) such that either \( y = x \) or \( y \in T_x \cdot Z_1 \).

It is easy to see that \( S \) is a separable Hausdorff space. By a slight modification of Jones’ argument, it may be shown that \( S \) is normal. In his argument where he considers \( K \cdot M_1 \) to be countable, replace the set \( D_{1k} \) by a set \( Q_{1k} \) such that \( Q_{1k} = Q \cdot M_1 \) where \( Q \) is an open set in \( S \) such that (1) \( Q \) is countable, (2) \( Q \cdot M_1 \) is a closed subset of \( M_1 \), and (3) \( Q \cdot M_1 \cdot H = Q \cdot H = 0 \).

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It will now be shown that $S$ is not paracompact. Let $G$ denote an open covering of $S$ such that an open set $D \in G$ if and only if either

1. $D$ is a point of $Z_1$ or
2. there exists $p$ in $M_1$ such that $D \cdot M_1$ is the set of all points $x$ in $M_1$ such that $x < p$ in $\alpha$.

Now, suppose that there exists an open refinement $G_1$ of $G$ which is locally finite. Then at most a finite number of the elements of $G_1$ intersects $M_1$. Otherwise, there exists a point $p$ in $M_1$ and an infinite subcollection $G_2$ of $G_1$ such that

1. for $g$ in $G_2$ and $x$ in $M_1$ where $p \leq x$ in $\alpha$, $g \not\supset x$ and
2. if $x < p$ in $\alpha$, then there exists $g$ in $G_2$ such that $g \supset x$; and furthermore, at most a finite number of the elements of $G_2$ contains $x$. Thus, for an open set $D \supset p$, $D$ intersects infinitely many of the elements of $G_1$. This is contrary to the supposition that $G_1$ is locally finite. If at most a finite number of the elements of $G_1$ intersects $M_1$, then some element of $G_1$ contains uncountably many points of $M_1$. This is impossible since $G_1$ is a refinement of $G$. It follows that $S$ is not paracompact.

Observe that $S$ is not a semi-metric topological space. Thus, the following questions arise naturally.

1. Is a normal separable semi-metric topological space paracompact?
2. Is a normal semi-metric topological space paracompact?

If the answer to (1) is "no," then it follows from work of F. B. Jones [3] that the continuum hypothesis is false. An affirmative answer to (1) would free an important result of Jones' from the continuum hypothesis.

Bibliography


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