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CONJUGATES IN DIVISION RINGS

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In this note we prove the

Theorem. *If in a division ring D an element $a\in D$ has only a finite number of conjugates in D then it has only one conjugate, that is, a is in Z, the center of D.*

This theorem, of course, generalizes the famous theorem of Wedderburn which asserts that a finite division ring is a commutative field; however, since Wedderburn's theorem is used in the proof it does not yield a new proof of the result of Wedderburn. We also exhibit two corollaries to the theorem which may be of some independent interest; the second of these extends the result that a polynomial over a field having more roots than its degree in some extension field must be identically zero to a suitable analogue when the roots lie in a division ring.

Proof of the theorem. We use the following convention throughout: if K is a division ring then K' will be the group of its nonzero elements under the multiplication of K.

Let $a\in D$ have a finite number of conjugates in D. Thus if $N=\{x\in D \mid xa=ax\}$ then N is a subdivision ring of D; moreover N' is of finite index in D'. Thus N' has a finite number of conjugates in D'. Consequently N has a finite number of conjugates in D, say $N=N_1$, $N_2, \cdots, N_k$; of course these N_i's are subdivision rings of D. Since the N_i''s are all of finite index in D' and there are a finite number of them, their intersection, T', is also of finite index in D'; in addition T' is normal in D'. Thus T, the intersection of the N_i is a subdivision

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ring of D invariant under all the inner automorphisms of D. By the Brauer-Cartan-Hua theorem [1] either $T=D$ or $T\subset Z$, the center of D. If $T=D$, then $N=D$ and so a is in Z. So we consider the second possibility, namely $T\subset Z$. But since T' is of finite index in D', the fact that $T\subset Z$ implies that Z' is of finite index in D'.

If Z is a finite field then since Z' is of finite index in D' it follows that D is a finite division ring, and so is commutative by Wedderburn's theorem.

So we suppose that Z has an infinite number of elements. Consider the elements $a_0=a$, $a_1=a+z_1, \cdots, a_n=a+z_n, \cdots$ where the z_i are an infinite number of distinct elements of Z. Since the index of Z' in D' is finite, for some $z_i \neq z_j$, a_i and a_j must be in the same coset of Z'; that is $a+z_i=z(a+z_j)$ where $z\in Z$. Since $z_i\neq z_j$, z can not be equal to 1; but then $(1-z)a=zz_j-z_i$ and so is in Z. Since $1-z$ is in Z and is not 0 it has an inverse in Z, from which we deduce that $a\in Z$, proving the theorem.

COROLLARY 1. *Let D be a division ring with center Z and suppose that $p(x)=\alpha_0x^n+\alpha_1x^{n-1}+\cdots+\alpha_n$ where the α_i are in Z, has one root in D which is outside of Z. Then $p(x)$ has an infinite number of roots in D.*

PROOF. Let $a\in D$, $a\notin Z$ be a root of $p(x)$; then all the conjugates of a in D are also roots; since $a\notin Z$ it has an infinite number of conjugates, proving the theorem.

COROLLARY 2. *Let D be a division ring with center Z and suppose that $p(x)$ is a polynomial of degree n with coefficients in Z. If $p(x)$ has $n+1$ roots in D then it has an infinite number of roots in D.*

PROOF. $p(x)$ has at most n roots in Z since Z is a field, thus since it has $n+1$ roots in D, one of these roots must fall outside Z, so the corollary reduces to Corollary 1.

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