BASIS STRUCTURE OF MODULES

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1. **Introduction.** Elementary divisor theory over the rational integers is classical; a treatment over an arbitrary algebraic number field can be found in Steinitz [**8**]. In this paper we are motivated by the following interpretation of Chevalley [**4**] and Asano [**2**]: if L_1 and L_2 are finite modules in a vector space V over a Dedekind ring $\mathfrak{o}$ with quotient field F, then L_1 and L_2 have a minimal basis in common. Simple examples show that in general this is not true for more than two modules L_1 and L_2, but in Theorem 4.2 we establish conditions under which every module in V has a minimal basis in common with a given set of modules $L_0, \cdots, L_m$. The basis structure of modules under a direct sum decomposition of V is examined in Theorem 4.3.

The results outlined above are proved over any valuation ring $\mathfrak{o}$: that they can be extended to an arbitrary Dedekind ring is a consequence of Theorem 5.4 which reduces the problem to a local one.

2. **Preliminaries.** Throughout this paper $\mathfrak{o}$ will denote an integral domain with a unit and with quotient field F; further restrictions will be placed on $\mathfrak{o}$ in the following paragraphs. The modules L considered will be torsion-free ($\alpha X=0$ with $\alpha\in\mathfrak{o}$, $X\in L$, implies $\alpha=0$ or $X=0$), finitely generated, and of finite rank; hence [**2**; **4**; **6**] each L has a replica in a finite dimensional F-space V; we will always assume that the modules under consideration are contained in such a vector space V; n will denote the dimension of this vector space; the subspace of V that is spanned by L will be written $F\cdot L$ and we have rank $L=\dim F\cdot L\leqq\dim V=n$.

In addition all modules will have at least one minimal basis in V—that is, a basis $\{\xi\}$ for V and (possibly zero) ideals $\mathfrak{a}_i$ in F such that $L=\mathfrak{a}_1\xi_1+\cdots+\mathfrak{a}_n\xi_n$; such a basis is known to exist for finitely generated modules in the following case:

THEOREM 2.1. *Let F be such that every finitely generated nonzero $\mathfrak{o}$-ideal is invertible. If L is a finitely generated $\mathfrak{o}$-module in V and if $\xi_1, \cdots, \xi_n$ is any basis for V, then there exist (finitely generated) ideals $\mathfrak{a}_i$ and a basis $\eta_\lambda=c_{1\lambda}\xi_1+\cdots+c_{\lambda-1\lambda}\xi_{\lambda-1}+\xi_\lambda$ for V such that $L=\sum\mathfrak{a}_\lambda\eta_\lambda$.*

This is proved in Satz 1 of [**2**] and the proof found there, though given for Dedekind rings, applies to fields of the above type; it is not necessary to assume that rank $L=n$. Note that the number of nonzero $\mathfrak{a}_i$ is equal to the rank of L. Compare the proofs in [**4**; **6**].

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If we say that a result $P(n, m)$, where n and m are non-negative integers, is proved by double induction we mean

(2.1) $\quad P(0, m)$ and $P(n, 0)$ hold for all n and m,

(2.2) $\quad P(n, m-1)$ and $P(n-1, m) \Rightarrow P(n, m)$.

One notational matter: we will denote the running indices in a summation by Greek letters, fixed indices by Roman or Gothic letters.

3. **Minimal bases for** L. In this section we investigate which bases in V qualify as minimal bases for L. The ring $\mathfrak{o}$ is quite general but it is assumed that L is a direct sum of $\mathfrak{o}$-ideals of F,

$$L = \mathfrak{a}_1\xi_1 + \cdots + \mathfrak{a}_n\xi_n. \tag{3.1}$$

Let us introduce the $\mathfrak{o}$-module

$$M = \mathfrak{b}_1\eta_1 + \cdots + \mathfrak{b}_n\eta_n, \tag{3.2}$$

the $\mathfrak{b}_i$ being $\mathfrak{o}$-ideals and $\{\eta\}$ being another basis for V connected with $\{\xi\}$ by the equations

$$\eta_j = \sum a_{\lambda j}\xi_\lambda, \qquad \xi_j = \sum b_{\lambda j}\eta_\lambda. \tag{3.3}$$

LEMMA 3.1. *$L \subseteq M$ if and only if $b_{ij}\mathfrak{a}_j \subseteq \mathfrak{b}_i$; $L = M$ if and only if $b_{ij}\mathfrak{a}_j \subseteq \mathfrak{b}_i$ and $a_{ij}\mathfrak{b}_j \subseteq \mathfrak{a}_i$; for all i and j.*

PROOF. It suffices to prove the first part of the lemma. Substituting (3.3) in (3.1) we see that $L \subseteq M$ is equivalent to $\sum \mathfrak{a}_\mu(\sum b_{\lambda\mu}\eta_\lambda) \subseteq \sum \mathfrak{b}_\lambda\eta_\lambda$; that is, $\mathfrak{a}_j b_{ij} \subseteq \mathfrak{b}_i$ for all i and j.

LEMMA 3.2. *The basis $\{\eta\}$ is a minimal basis for L if and only if $a_{ij}b_{jk}\mathfrak{a}_k \subseteq \mathfrak{a}_i$ for all i, j, k.*

PROOF. The necessity is an immediate consequence of Lemma 3.1: for if $a_{ij} \neq 0$, then $b_{jk}\mathfrak{a}_k \subseteq \mathfrak{b}_j \subseteq \mathfrak{a}_i a_{ij}^{-1}$. To prove the sufficiency we introduce the ideals $\mathfrak{b}_i = \sum b_{i\lambda}\mathfrak{a}_\lambda$ and then define M by means of equation (3.2). Then $L = M$ (again by Lemma 3.1) and so $\{\eta\}$ is a minimal basis for L.

LEMMA 3.3. *If $L = M$ then $a_{ij}\mathfrak{b}_j \subseteq \mathfrak{a}_i$ and* $\det\{a_{ij}\}\prod \mathfrak{b}_\lambda = \prod \mathfrak{a}_\lambda$; *conversely, if these conditions are satisfied and if the $\mathfrak{b}_i$ are invertible, then $L = M$.*

PROOF. *Necessity.* It follows from Lemma 3.1 and the expansion $\det\{a_{ij}\} = \sum \pm a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ that $\det\{a_{ij}\}\prod \mathfrak{b}_\lambda \subseteq \prod \mathfrak{a}_\lambda$. Similarly $\prod \mathfrak{b}_\lambda \supseteq \det\{b_{ij}\}\prod \mathfrak{a}_\lambda$. The remark $\det\{a_{ij}\} = (\det\{b_{ij}\})^{-1}$ completes the proof of the necessity.

Sufficiency. Let a'_{rs} denote the cofactor of a_{rs}. Then $a'_{rs} = \sum \pm a_{1\alpha} \cdots a_{n\omega}$, the first subscript avoiding r, the second s. Hence $a'_{rs}\mathfrak{a}_r\mathfrak{b}_s^{-1} \subseteq \prod \mathfrak{a}_\lambda\mathfrak{b}_\lambda^{-1} = \det\{a_{ij}\}$ and so $b_{sr}\mathfrak{a}_r \subseteq \mathfrak{b}_s$. The result follows from Lemma 3.1.

REMARK. It follows from the previous lemma that if L and M are isomorphic, then $\prod \mathfrak{a}_\lambda$ and $\prod \mathfrak{b}_\lambda$ are in the same ideal class. Compare [6].

LEMMA 3.4. *Suppose that $L = M$ and let the bases $\{\xi\}$ and $\{\eta\}$ have the property that $F\xi_{m+1} + \cdots + F\xi_n = F\eta_{m+1} + \cdots + F\eta_n$ for some m, $0 \leqq m \leqq n-1$. Then $\mathfrak{a}_{m+1}\xi_{m+1} + \cdots + \mathfrak{a}_n\xi_n = \mathfrak{b}_{m+1}\eta_{m+1} + \cdots + \mathfrak{b}_n\eta_n$.*

PROOF. The given restrictions ensure that all entries a_{ij} and b_{ij} are zero if $i \leqq m$ and $j \geqq m+1$. Then $\{a_{ij}\}_{i,j\geqq m+1}$ has $\{b_{ij}\}_{i,j\geqq m+1}$ as its inverse. But a_{ij} and b_{ij} satisfy the conditions of Lemma 3.1 for all i, j and hence for $i, j \geqq m+1$. Q.E.D.

We conclude this section with some remarks on the multiplication of modules by nonzero scalars $t \in F$. (a) $t \cdot L$ is again an $\mathfrak{o}$-module in V. (b) If $\{L_\lambda\}$ is a collection of $\mathfrak{o}$-modules in V and if $\{t_\lambda\}$ is a corresponding collection of scalars, then the L_λ have a basis $\{\xi\}$ in common if and only if all the $t_\lambda \cdot L_\lambda$ have the same minimal basis in common. (c) If $\mathfrak{o}$ is a valuation ring or a Dedekind ring, there are nonzero scalars t_1, t_2 such that $t_1 \cdot L \subseteq M$ and $t_2 \cdot L \supseteq M$, provided that L and M are both of rank n (Lemma 3.1).

4. **Local theory.** The ring $\mathfrak{o}$ is now taken to be a valuation ring with quotient field F: $\mathfrak{o}$ is an integral domain such that for any α in F, either $\alpha \in \mathfrak{o}$ or $\alpha^{-1} \in \mathfrak{o}$. Associated with $\mathfrak{o}$ is a valuation $|\alpha|$, $\alpha \in F$, with values in an ordered multiplicative group [1; 7]. We denote the maximal prime ideal in $\mathfrak{o}$ with the letter $\mathfrak{p}$. Observe that all finitely generated $\mathfrak{o}$-ideals are principal and hence invertible. All modules will be finitely generated and hence of the form of equation (3.1) with the $\mathfrak{a}_i$ principal by Theorem 2.1. Hence there is a basis for L in which

$$L = (p_1)\xi_1 + \cdots + (p_n)\xi_n, \qquad p_i \in F; \tag{4.1}$$

putting $\xi_i' = p_i\xi_i$ we see that we can take $L = \sum \mathfrak{o}\xi_\lambda'$ if desired. Furthermore, if $L = \sum \mathfrak{b}_\lambda\eta_\lambda$ is another representation in a minimal basis $\{\eta\}$, then the $\mathfrak{b}_\lambda$ must all be principal.

LEMMA 4.1. *Let $L_j = \sum \mathfrak{a}_{\lambda j}\xi_\lambda = \sum \mathfrak{b}_{\lambda j}\eta_\lambda$ be modules in V, $0 \leqq j \leqq m$, and let $L_0 = \sum \mathfrak{o}\xi_\lambda = \sum \mathfrak{o}\eta_\lambda$. Then there is a reordering of the $\{\eta\}$ in which $\mathfrak{a}_{ij} = \mathfrak{b}_{ij}$ for all i, j.*

PROOF. By induction to n. In the notation of (3.3), $\{a_{ij}\}$ must be

unimodular. Now $\sum a_{1\lambda}b_{\lambda 1}=1$; therefore there is a k for which $|a_{1k}|=1=|b_{k1}|=|a'_{1k}|$. Interchanging η_1 and η_k we have

$$(4.2) \qquad |a_{11}| = 1 = |b_{11}|.$$

Then $a_{11}\mathfrak{b}_{1j}\subseteq\mathfrak{a}_{1j}$ by Lemma 3.1 and so $\mathfrak{b}_{1j}\subseteq\mathfrak{a}_{1j}$. Similarly $\mathfrak{a}_{1j}\subseteq\mathfrak{b}_{1j}$. Hence

$$(4.3) \qquad \mathfrak{a}_{1j} = \mathfrak{b}_{1j} \qquad \text{for all } j.$$

Now $a_{1\lambda}\mathfrak{b}_{\lambda j}\subseteq\mathfrak{a}_{1j}$ together with (4.2) and (4.3) implies that $a_{1\lambda}a_{11}^{-1}\mathfrak{b}_{\lambda j}\subseteq\mathfrak{b}_{1j}$ and therefore $\sum\mathfrak{b}_{\lambda j}\eta_\lambda = \mathfrak{b}_{1j}\eta_1+\sum\mathfrak{b}_{\lambda j}(\eta_\lambda - a_{1\lambda}a_{11}^{-1}\eta_1)$. But $(\eta_\lambda-a_{1\lambda}a_{11}^{-1}\eta_1)\in F\xi_2+\cdots+F\xi_n$ for $\lambda\geqq 2$; hence $\sum\mathfrak{b}_{\lambda j}(\eta_\lambda-a_{1\lambda}a_{11}^{-1}\eta_1)=\sum\mathfrak{a}_{\lambda j}\xi_\lambda$, with $\lambda\geqq 2$, in virtue of Lemma 3.4. Induction completes the proof.

NOTE. In the case $m=1$ the previous lemma expresses the "uniqueness of the elementary divisors."

THEOREM 4.2. *Let L_j, $0\leqq j\leqq m$, be given rank n modules in V. Then every rank n module K in V has a mininal basis $\{\xi^K\}$ which is also a minimal basis for all the L_j if and only if either $tL_j\subseteq L_i$ or $tL_j\supseteq L_i$ holds for all i, j and all scalars $t\in F$.*

PROOF. *Sufficiency.* By double induction. (2.1) is true by the elementary divisor theorem which holds for valuation rings. We must prove (2.2).

By the induction and (4.1) we can write $L_0,\cdots,L_{m-1}$ and L_m in a common minimal basis such that $L_0=\sum\mathfrak{o}\xi_\lambda$, $L_j=\sum(p_{\lambda j})\xi_\lambda$, the $(p_{\lambda j})$ being principal ideals in F. By considering t_jL_j if necessary we can make the further assumption that $L_j\subseteq L_0$ and $L_0\neq\pi^{-1}L_j\supseteq L_0$ for any $\pi\in\mathfrak{p}$, $0\leqq j\leqq m$. Reordering $L_1,\cdots,L_m$ we obtain $L_0\supseteq L_1\supseteq L_2\cdots\supseteq L_m$; hence

$$(4.4) \qquad |p_{\lambda m}| \leqq |p_{\lambda m-1}| \leqq \cdots \leqq |p_{\lambda 0}| = 1, \qquad (p_{\lambda m}) = \mathfrak{p} \text{ or } \mathfrak{o}.$$

Let $\gamma(\lambda)$ denote the number of p's with absolute value 1 in this inequality.

Let K be of the form $\sum\mathfrak{o}\eta_\nu$ with $\xi_i=\sum b_{\mu i}\eta_\mu$. By rearranging $\{\xi\}$ and $\{\eta\}$ we can assume that

$$(4.5) \quad |b_{11}| = \max|b_{i\lambda}|, \qquad |b_{1\lambda}| = |b_{11}| \Rightarrow \gamma(1)\geqq\gamma(\lambda)'.$$

Now define $\mathrm{H}_1=\sum b_{\lambda 1}b_{11}^{-1}\eta_\lambda$ and $\Xi_\lambda=(\xi_\lambda-b_{1\lambda}b_{11}^{-1}\xi_1)$. Then $(p_{1j})\xi_1+(p_{\lambda j})\xi_\lambda=(p_{1j})\xi_1+(p_{\lambda j})\Xi_\lambda$, since $|b_{1\lambda}b_{11}^{-1}|<1$ implies $|p_{\lambda j}b_{1\lambda}b_{11}^{-1}|\leqq|p_{1j}|$, while if $|b_{1\lambda}b_{11}^{-1}|=1$, then $\gamma(\lambda)\leqq\gamma(1)$ implies $|p_{\lambda j}|\leqq|p_{1j}|$. Hence $L_j=(p_{1j})\xi_1+L_j'$ where $L_j'=\sum(p_{\lambda j})\Xi_\lambda$. Since $|b_{\lambda 1}b_{11}^{-1}|\leqq 1$, we must have $K=\mathfrak{o}\mathrm{H}_1+K'$ where $K'=\mathfrak{o}\eta_2+\cdots+\mathfrak{o}\eta_n$. But $F\xi_1=F\mathrm{H}_1$

and $\sum F\Xi_\lambda = F\eta_2 + \cdots + F\eta_n$ by definition of H_1 and Ξ_λ. In addition $L'_{j+1} \subseteq L'_j$ and $L'_{j+1} \supseteq \mathfrak{p}L'_j$ by (4.4). Applying the induction again we see that L'_j, $0 \leqq j \leqq m$, and K' have a minimal basis $\{\omega\}$ in common. Hence $\xi_1, \omega_2, \cdots, \omega_n$ is a minimal basis for L_j, $0 \leqq j \leqq m$, and K. This proves (2.2) and hence establishes the induction.

Necessity. Given any pair L_i, L_j there is a minimal basis $\{\xi\}$ such that $L_i = \sum \mathfrak{o}\xi_\lambda$ and $L_j = \sum \mathfrak{b}_\lambda \xi_\lambda$ by the elementary divisor theorem. Hence if the given conditions on the L_j, $0 \leqq j \leqq m$, do not hold there must be a pair, L_0 and L_1 say, such that $L_0 = \sum \mathfrak{o}\xi_\lambda$ and $tL_1 = (p)\xi_1 + \mathfrak{o}\xi_2 + (p_3)\xi_3 + \cdots + (p_n)\xi_n$ with $p_\lambda \in \mathfrak{o}$ and $|p| < |\pi| < 1$, for some $\pi \in \mathfrak{p}$. By a change of notation let us refer to tL_1 as L_1. Now define a new basis $\{\eta\}$ for V by the equations $\{\eta\}B = \{\xi\}$ where

$$\xi_1 = \eta_1, \qquad \xi_2 = \pi\eta_1 + p\eta_2, \qquad \xi_j = \eta_j,\ j > 2. \tag{4.6}$$

Put $K = \sum \mathfrak{o}\eta_\lambda$. Contention: L_0, L_1 and K do not have a common minimal basis.

Suppose that the contrary is true and let their common minimal basis be

$$\{\zeta\} = \{\xi\}C = \{\eta\}BC. \tag{4.7}$$

Put $C = \{c_{ij}\}$ and $C^{-1} = \{d_{ij}\}$. Then the first row and kth column of BC has entry $(c_{1k} + \pi c_{2k})$ while the kth row and second column of $(BC)^{-1}$ has entry $(d_{k2} - \pi d_{k1})p^{-1}$. Now apply Lemma 3.2 (i) to L_0: $c_{ik}d_{kj} \in \mathfrak{o}$ all i, j, k; (ii) to L_1: $c_{1k}d_{k2} \in (p)$ for all k; (iii) to K: $(c_{1k} + \pi c_{2k})(d_{k2} - \pi d_{k1})p^{-1} \in \mathfrak{o}$. Then these three conditions yield (iv): $c_{1k}d_{k1} - c_{2k}d_{k2} \in \mathfrak{p}$. Now $(c_{1k}d_{k1})(c_{2k}d_{k2}) \in (p)$ by (i) and (ii) and so either $c_{1k}d_{k1} \in \mathfrak{p}$ or $c_{2k}d_{k2} \in \mathfrak{p}$ and hence both must be in $\mathfrak{p}$ by (iv). Hence $1 = \sum c_{1\lambda}d_{\lambda 1} \in \mathfrak{p}$ and this is a contradiction. Q.E.D.

REMARK. There is a different way of stating Theorem 4.2—by assuming that $FL_0 = V$ but dropping the conditions on the ranks of the L_j, $0 \leqq j \leqq m$, and on K.

Sufficiency. It is easy to see that the conditions of the theorem guarantee that all L_j must be of the same rank and therefore span V. If in addition rank $K = n$, the proof is as before. If this is not so, write $K = 0 \cdot \eta_1 + \mathfrak{a}_2\eta_2 + \cdots + \mathfrak{a}_n\eta_n$; instead of (4.5) we can arrange to have

$$|b_{11}| = \max |b_{1\lambda}|, \qquad |b_{1\lambda}| = |b_{11}| \Rightarrow \gamma(1) \geqq \gamma(\lambda)$$

by permuting the $\{\xi\}$ only. The rest of the proof is as before.

Necessity. If all L_j, $0 \leqq j \leqq m$, are of rank n the previous proof is valid. If not, we can take $L_0 = \sum \mathfrak{o}\xi_\lambda$ and $L_1 = 0 \cdot \xi_1 + \mathfrak{o}\xi_2 + \mathfrak{a}_3\xi_3 + \cdots + \mathfrak{a}_n\xi_n$ by the elementary divisor theorem; put $K = \mathfrak{o}(\xi_1 + \pi^{-1}\xi_2)$ where

$\pi \in \mathfrak{p}$; then it is easy to see (e.g. by using Theorem 2.1 and Lemma 3.4) that L_0, L_1 and K do not have a common minimal basis.

THEOREM 4.3. *If L_j, $0 \leqq j \leqq m$, are modules in V having a minimal basis in common and if each L_j has a decomposition $L_j = L_j' + L_j''$ corresponding to a fixed direct sum decomposition $V = V' + V''$, then the L_j' (resp. L_j'') have a common minimal basis in V' (resp. V'').*

First we prove the following special case.

LEMMA. *The theorem is true if $FL_0 = V$.*

PROOF OF LEMMA. By double induction. (2.1) is true. We must prove (2.2). By the induction we have

$$(4.8) \quad L_j' = \sum_1^q (p_{\lambda j})\xi_\lambda \subseteq V', \qquad L_j'' = \sum_{q+1}^n (p_{\lambda j})\xi_\lambda \subseteq V'', \qquad 0 \leqq j \leqq m-1,$$

with $L_0 = \sum \mathfrak{o}\xi_\lambda$, that is $p_{\lambda 0} = 1$ for all λ. Let $\{\zeta\}$ be a minimal basis common to all the L_j, $0 \leqq j \leqq m$; there is no loss of generality in taking

$$(4.9) \quad L_0 = \sum \mathfrak{o}\zeta_\lambda, \quad L_j = \sum (\bar{p}_{\lambda j})\zeta_\lambda, \quad L_m = \mathfrak{o}\zeta_1 + \sum_2^n (\bar{p}_{\lambda m})\zeta_\lambda, \qquad \bar{p}_{\lambda m} \in \mathfrak{o}.$$

If $\zeta_i = \sum c_{\lambda i}\xi_\lambda$, then $\{c_{ij}\}$ must be unimodular and we can permute $\{\xi\}$ as in Lemma 4.1 until $|c_{11}| = 1 = |d_{11}|$, interchange V' and V'' if necessary, and then permute $\zeta_2, \cdots, \zeta_n$ until

$$(4.10) \qquad \sum (p_{\lambda j})\xi_\lambda = L_j = \sum (p_{\lambda j})\zeta_\lambda, \qquad 0 \leqq j \leqq m-1,$$

$$(4.11) \qquad |c_{11}| = 1, \qquad c_{\lambda\mu}p_{\mu j} \in (p_{\lambda j}), \qquad 0 \leqq j \leqq m-1.$$

Note that with a corresponding change of notation, (4.8) and (4.9) are preserved.

Now put $\mathbf{Z}_1 = \zeta_1$, $\mathbf{Z}_\lambda = (\zeta_\lambda - c_{1\lambda}c_{11}^{-1}\zeta_1)$, $\Xi_1 = \sum_1^q c_{\lambda 1}c_{11}^{-1}\xi_\lambda$, $\Xi_\lambda = \xi_\lambda$ and then write ζ and ξ instead of $\mathbf{Z}$ and Ξ. In virtue of (4.11) and these substitutions, (4.8), (4.9) and (4.10) are still valid and

$$(4.12) \qquad \zeta_1 = c_{11}\xi_1 + \sum_{q+1}^n c_{\lambda 1}\xi_\lambda, \qquad \zeta_i = \sum_2^n c_{\lambda i}\xi_\lambda, \qquad i > 1.$$

It follows from the direct sum decomposition of L_m and (4.9) that $\sum_{q+1}^n c_{\lambda 1}\xi_\lambda \in L_m$ and $c_{11}\xi_1 \in L_m$. Hence we see from (4.9) and (4.12) that $L_m = \sum_2^n (\bar{p}_{\lambda m})\zeta_\lambda + \mathfrak{o}\xi_1$. Now $\sum_2^n (\bar{p}_{\lambda m})\zeta_\lambda$ and $\sum_2^n (p_{\lambda j})\xi_\lambda$, $0 \leqq j \leqq m-1$, have $\zeta_2, \cdots, \zeta_n$ as a common minimal basis in virtue of Lemma 3.4 and (4.12). Contention: there is a direct sum decomposition of $\sum_2^n (\bar{p}_{\lambda m})\zeta_\lambda$ corresponding to $\sum_2^q F\xi_\lambda \oplus \sum_{q+1}^n F\xi_\lambda$: for $L_m = L_m' + L_m''$;

and $L'_m = \mathfrak{o}\xi_1 + \sum_2^q \mathfrak{a}_\lambda \eta_\lambda$ with $\eta_\lambda = \alpha_{1\lambda}\xi_1 + \cdots + \xi_\lambda$ in virtue of Theorem 2.1; but $\mathfrak{a}_\lambda \alpha_{1\lambda} \subseteq \mathfrak{o}$ since $L_m \subseteq L_0$; hence we can assume that $\alpha_{1\lambda} = 0$, $2 \leqq \lambda \leqq q$; then by Lemma 3.4, $\sum_2^n (\bar{p}_{\lambda m})\zeta_\lambda = \sum_2^q \mathfrak{a}_\lambda \eta_\lambda \oplus L''_m$. This proves the contention. Induction completes the proof. Q.E.D.

PROOF OF THEOREM. By double induction. By applying the elementary divisor theorem to L'_0 (resp. L''_0) and L'_1 (resp. L''_1) we can write

$$(4.13)\qquad \begin{aligned} L_0 &= \mathfrak{o}\xi_1 + \cdots + \mathfrak{o}\,\xi_r + \cdots + \mathfrak{o}\,\xi_t, \\ L_1 &= \qquad\qquad\qquad \mathfrak{p}_r\xi_r + \cdots + \mathfrak{p}_t\xi_t + \mathfrak{o}\xi_{t+1} + \cdots + \mathfrak{o}\xi_s \end{aligned}$$

with $s \leqq n$ and $\{\xi\}$ such that either $\xi_\lambda \in V'$ or $\xi_\lambda \in V''$; there is no loss of generality in assuming that $\mathfrak{p}_\lambda \subseteq \mathfrak{o}$, $r \leqq \lambda \leqq t$. On the other hand an easy computation shows that we can assume that equations (4.13) are true if we replace $\{\xi\}$ by $\{\zeta\}$ where $\{\zeta\}$ is a minimal basis common to all L_j, $0 \leqq j \leqq m$. Define K invariantly as the smallest $\mathfrak{o}$-module containing L_0 and L_1. Then

$$\sum_1^s \mathfrak{o}\xi_\lambda = K = \sum_1^s \mathfrak{o}\zeta_\lambda.$$

CASE I. $F \cdot K = V$. Then $K, L_0, L_1, \cdots, L_m$ satisfy the conditions of the lemma; the theorem therefore holds for these modules; hence it holds for L_j, $0 \leqq j \leqq m$.

CASE II. $F \cdot K \neq V$. By the induction $K, L_2, L_3, \cdots, L_m$ have a common basis $\{\eta\}$ such that $K = \sum_1^s \mathfrak{o}\eta_\lambda$ and either $\eta_\lambda \in V'$ or $\eta_\lambda \in V''$ for $1 \leqq \lambda \leqq n$. By means of Lemma 3.4 we see that L_0, L_1 and $\sum_1^s \mathfrak{a}_{\lambda j}\eta_\lambda$, $2 \leqq j \leqq m$, must have $\zeta_1, \cdots, \zeta_s$ as a common minimal basis and hence have a basis $\omega_1, \cdots, \omega_s$ with the required property. Then $\omega_1, \cdots, \omega_s$, $\eta_{s+1}, \cdots, \eta_n$ is the basis required. This completes the proof of the theorem.

LEMMA 4.4. *Let L_j be rank n modules in V and let X be any vector in L_0. Then there is a nonzero α (independent of X) such that if $\{\xi\}$ is any basis for all L_j with $L_0 = \sum \mathfrak{o}\xi_\lambda$, then so is $\xi_1, \cdots, \xi_{i-1}, \xi_i + \alpha' X, \xi_{i+1}, \cdots, \xi_n$ where $|\alpha'| \leqq |\alpha|$.*

PROOF. Write $L_j = \sum \mathfrak{a}_{\lambda j}\xi_\lambda$ and $X = \sum \alpha_\lambda \xi_\lambda$ where $\alpha_\lambda \in \mathfrak{o}$; choose $\alpha \in \mathfrak{p}$ such that $\alpha \in \mathfrak{a}_{\lambda j}\mathfrak{a}_{\mu j}^{-1}$ for all λ, μ, j. Then

$$L_j = \mathfrak{a}_{1j}\xi_1 + \cdots + \mathfrak{a}_{ij}(\xi_i + \alpha' X) + \cdots + \mathfrak{a}_{nj}\xi_n.$$

5. **Dedekind rings.** In this section we assume that classical ideal theory holds in $F/\mathfrak{o}$, that is to say the ideals with bounded denominator form a group under multiplication. We denote by $\mathfrak{o}_\mathfrak{p}$ the valuation ring corresponding to the prime ideal $\mathfrak{p}$; the same letter $\mathfrak{p}$ will be used

for the maximal prime ideal in $\mathfrak{o}_{\mathfrak{p}}$. The $\mathfrak{o}_{\mathfrak{p}}$-ideals in F are all powers of $\mathfrak{p}$ and the value group is discrete and real. If $\mathfrak{a}=\mathfrak{p}^{\nu}\cdots$ is a factorization of the $\mathfrak{o}$-ideal $\mathfrak{a}$ we define $\mathfrak{p}(\mathfrak{a})$ as the $\mathfrak{o}_{\mathfrak{p}}$-ideal $\mathfrak{p}^{\nu}$. All modules will be finitely generated and hence of the form (3.1) with the $\mathfrak{a}_i$ of bounded denominator and invertible. Note that if $\xi_1, \cdots, \xi_n$ is any basis for V, then by Theorem 2.1 there is a minimal basis $\eta_1, \cdots, \eta_n$ for L such that

$$\eta_i = c_{1i}\xi_1 + \cdots + c_{i-1i}\xi_{i-1} + \xi_i, \qquad c_{ij} \in F. \tag{5.1}$$

If $L=\sum \mathfrak{a}_\lambda \xi_\lambda$ put $\mathfrak{p}(L)=\sum \mathfrak{p}(\mathfrak{a}_\lambda)\xi_\lambda$: that this is well-defined is an immediate consequence of Lemma 3.1. By taking $\xi_\lambda' = \alpha_\lambda \xi_\lambda$ instead of ξ_λ it is always possible to choose a basis for L such that $\mathfrak{p}(L)=\sum \mathfrak{o}_{\mathfrak{p}}\xi_\lambda'$ at a given finite set of primes $\mathfrak{p}$ provided that rank $L=n$. By means of Lemma 3.1 and the unique factorization into primes it is easy to prove the following result.

LEMMA 5.1. *$L=M$ if and only if $\mathfrak{p}(L)=\mathfrak{p}(M)$ at all $\mathfrak{p}$. In any case, if $F\cdot L=F\cdot M$, then $\mathfrak{p}(L)=\mathfrak{p}(M)$ holds at all but a finite number of primes $\mathfrak{p}$.*

LEMMA 5.2. *Let L_j, $0\leqq j\leqq m$, be rank n modules in V and let W be an r dimensional subspace of V. If $\mathfrak{p}(L_j)$, $0\leqq j\leqq m$, have a common minimal basis $\{\eta^{\mathfrak{p}}\}$ such that $\eta_1^{\mathfrak{p}}, \cdots, \eta_r^{\mathfrak{p}} \in W$ at each $\mathfrak{p}$ in a finite set of primes S, then there is a basis $\{\zeta\}$ for L_0 which is also a basis for all the $\mathfrak{p}(L_j)$, $\mathfrak{p}\in S$, and such that $\zeta_1, \cdots, \zeta_r \in W$.*

PROOF. By induction on n. All bases (whether defined or derived) that appear in the proof will be such that their first r elements span W. Let us write $L_0=\sum \mathfrak{a}_{\lambda 0}\xi_\lambda$ and $\eta_i^{\mathfrak{p}}=\sum a_{\lambda i}^{\mathfrak{p}}\xi_\lambda$; we can assume that $\mathfrak{p}(\mathfrak{a}_{\lambda 0})=\mathfrak{o}_{\mathfrak{p}}$ and $\mathfrak{p}(L_0)=\sum \mathfrak{o}_{\mathfrak{p}}\eta_\lambda^{\mathfrak{p}}$ at all $\mathfrak{p}\in S$. Apply the approximation theorem to the entries of the matrices $\{a_{ij}^{\mathfrak{p}}\}$ in such a way that a_{ij} approximates to $a_{ij}^{\mathfrak{p}}$ with $a_{ij}=0$ whenever $a_{ij}^{\mathfrak{p}}=0$ is true for all $\mathfrak{p}\in S$; then in virtue of Lemma 4.4 a sufficiently high approximation yields a basis $\eta_j=\sum a_{\lambda j}\xi_\lambda$ such that

$$\mathfrak{p}(L_0) = \sum \mathfrak{o}_{\mathfrak{p}}\xi_\lambda, \qquad \mathfrak{p}(L_j) = \sum \mathfrak{p}_{\lambda j}\eta_\lambda, \qquad \text{all } \mathfrak{p} \in S. \tag{5.2}$$

Now write each $L_j=\sum \mathfrak{a}_{\lambda j}\xi_{\lambda j}'$ with $\xi_{\lambda j}' \in F\eta_1 + \cdots + F\eta_\lambda$, by (5.1). Then all the modules $\sum_1^{n-1} \mathfrak{a}_{\lambda j}\xi_{\lambda j}'$ have $\eta_1, \cdots, \eta_{n-1}$ as a common minimal basis at each $\mathfrak{p}\in S$, in virtue of Lemma 3.4; by the induction there is a basis (and we continue to call it $\{\xi\}$) for L_0 such that $\eta_1, \cdots, \eta_n$ is a basis for the $\mathfrak{p}(L_j)$, $\mathfrak{p}\in S$, where $\eta_i=\xi_i$ for all $i\leqq n-1$; we can assume that

$$\sum \mathfrak{o}_{\mathfrak{p}}\xi_\lambda = \mathfrak{p}(L_0) = \sum \mathfrak{o}_{\mathfrak{p}}\eta_\lambda, \qquad \mathfrak{p} \in S. \tag{5.3}$$

Suppose that $\eta_n = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$; then $|\alpha_i| \leqq 1$ and $|\alpha_n| = 1$ at all $\mathfrak{p} \in S$; by suitably scaling η_n we can make the further assumptions

$$(5.4) \qquad \alpha_i \in \mathfrak{o}, \qquad \alpha_i \in \mathfrak{a}_{i0}\mathfrak{a}_{n0}^{-1} \qquad \text{for all } i.$$

Let $\alpha, \alpha' \in \mathfrak{o}$ be such that $|\alpha|$, $|\alpha'|$ are arbitrarily small at $\mathfrak{p} \in S$, but either $|\alpha| = 1$ or $|\alpha'| = 1$ outside S; then g.c.d. $(\alpha, \alpha', \alpha_n) = \mathfrak{o}$ and there must exist $\beta, \beta', \gamma \in \mathfrak{o}$ such that $\alpha\beta + \alpha'\beta' + \gamma\alpha_n = 1$; clearly γ must be a unit at all $\mathfrak{p} \in S$. Define

$$(5.5) \qquad \zeta_i = \xi_i,\ i < n, \qquad \zeta_n = \alpha\beta\xi_n + \alpha'\beta'\xi_n + \gamma\eta_n.$$

Then $\zeta_n = \sum_1^{n-1} \gamma\alpha_\lambda\xi_\lambda + \xi_n$ and so $\{\zeta\}$ is a basis for L_0. Since $|\alpha|$, $|\alpha'|$ can be made arbitrarily small at $\mathfrak{p} \in S$, we have $\{\zeta\}$ as a basis for $\mathfrak{p}(L_j)$, $\mathfrak{p} \in S$, by Lemma 4.4. Q.E.D.

LEMMA 5.3. *If the assumptions of the previous lemma are satisfied at ALL primes* $\mathfrak{p}$, *then there is a common basis* $\{\zeta\}$ *for all* L_j, $0 \leqq j \leqq m$, *such that* $\zeta_1, \cdots, \zeta_r \in W$.

PROOF. Let S be a finite set of primes such that $\mathfrak{p}(L_0) = \mathfrak{p}(L_j)$ if $\mathfrak{p} \notin S$. Let $\{\zeta\}$ be a basis having the properties of Lemma 5.2; then $\mathfrak{p}(L_0) = \sum \mathfrak{p}_{\lambda 0}\zeta_\lambda = \mathfrak{p}(L_j)$ if $\mathfrak{p} \notin S$, and $\mathfrak{p}(L_j) = \sum \mathfrak{p}_{\lambda j}\zeta_\lambda$ if $\mathfrak{p} \in S$. Define

$$\mathfrak{a}_{\lambda j} = \prod_{\mathfrak{p} \in S} \mathfrak{p}_{\lambda j} \prod_{\mathfrak{p} \notin S} \mathfrak{p}_{\lambda 0}.$$

Then $L_j = \sum \mathfrak{a}_{\lambda j}\zeta_\lambda$ by Lemma 5.1.

THEOREM 5.4. *The modules* L_j *have a common minimal basis if and only if their local components* $\mathfrak{p}(L_j)$ *have a common minimal basis at all primes* $\mathfrak{p}$, $0 \leqq j \leqq m$.

PROOF. By double induction. By the elementary divisor theorem

$$L_0 = \mathfrak{a}_{10}\omega_1 + \cdots + \mathfrak{a}_{r0}\omega_r + \cdots + \mathfrak{a}_{q0}\omega_q,$$
$$L_1 = \qquad\qquad \mathfrak{a}_{r1}\omega_r + \cdots + \mathfrak{a}_{q1}\omega_q + \cdots + \mathfrak{a}_{s1}\omega_s.$$

We can assume that $\mathfrak{a}_{\lambda 1} \subseteq \mathfrak{a}_{\lambda 0}$, $r \leqq \lambda \leqq q$. Define K invariantly as the $\mathfrak{o}$-module generated by L_0 and L_1; then $K = \sum(\mathfrak{a}_{\lambda 0} + \mathfrak{a}_{\lambda 1})\omega_\lambda$.

CASE I. Rank $L_j = n$, $0 \leqq j \leqq m$. This follows from Lemma 5.3.

CASE II. Rank $L_0 < n$, rank $L_j = n$, $1 \leqq j \leqq m$. By Lemma 5.3 we can express L_j, $1 \leqq j \leqq m$, in a common $\{\xi\}$ with $\xi_1, \cdots, \xi_q$ spanning $F \cdot L_0$: $L_j = \sum \mathfrak{a}_{\lambda j}\xi_\lambda$, $1 \leqq j \leqq m$. Then L_0 and $\sum_1^q \mathfrak{a}_{\lambda j}\xi_\lambda$ have a common basis at all $\mathfrak{p}$ in virtue of Lemma 3.4 and so by the induction they must have a common basis $\zeta_1, \cdots, \zeta_q$ in the large. Then $\zeta_1, \cdots, \zeta_q$, $\xi_{s+1}, \cdots, \xi_n$ is the basis required.

CASE III. Rank $K=s<n$. Then if $\{\eta^{\mathfrak{p}}\}$ is a basis for $\mathfrak{p}(L_j)$, $0\leqq j\leqq m$, it must also be a basis for $\mathfrak{p}(K)$; therefore by the induction K, $L_2, \cdots, L_m$ have a common basis $\{\xi\}$; let us suppose that $\xi_1, \cdots, \xi_s$ spans $F\cdot K$; then L_0, L_1 and $\sum_1^s \mathfrak{a}_{\lambda j}\xi_\lambda$ have a common minimal basis at all $\mathfrak{p}$ in virtue of Lemma 3.4; hence they have a basis $\zeta_1, \cdots, \zeta_s$ in the large. Then $\zeta_1, \cdots, \zeta_s, \xi_{s+1}, \cdots, \xi_n$ is the basis required.

CASE IV. Rank $L_0<n$, rank $L_1<n$ and rank $K=s=n$. Then by (5.1) we can express each L_j, $2\leqq j\leqq m$, in a basis whose successive elements are in $F\omega_n$, $F\omega_n+F\omega_{n-1}$, etc.; by Lemma 3.4 and the inductive hypothesis we see that we can assume that $L_j=\sum_1^{r-1} \mathfrak{a}_{\lambda j}\omega'_{\lambda j} + \sum_r^n \mathfrak{a}_{\lambda j}\omega_\lambda$, for $0\leqq j\leqq m$.

Contention: we can arrange to have $\omega'_{\lambda j}\in FL_0$ when $2\leqq j\leqq m$. For each $\alpha\in\mathfrak{a}_{\lambda j}$ we have $\alpha\omega'_{\lambda j}=\alpha\omega_\lambda+\alpha\rho_{\lambda+1j}\omega_{\lambda+1}+\cdots+\cdots\in L_j$; at each $\mathfrak{p}$, $\mathfrak{p}(L_j)$ has a basis $\{\eta^{\mathfrak{p}}\}$ with $\eta_1^{\mathfrak{p}}, \cdots, \eta_q^{\mathfrak{p}}\in FL_0$, $\eta_r^{\mathfrak{p}}, \cdots, \eta_n^{\mathfrak{p}} \in FL_1$; and $\eta_\mu^{\mathfrak{p}}=\omega_\mu$ for $\mu\geqq r$, by Lemma 3.4, hence

$$\alpha\omega'_{\lambda j} = \beta_{1j}\eta_1^{\mathfrak{p}} + \cdots + \beta_{qj}\eta_q^{\mathfrak{p}} + \alpha\rho_{q+1j}\omega_{q+1} + \cdots + \alpha\rho_{nj}\omega_n;$$

hence $\alpha\rho_{\mu j}\omega_\mu\in\mathfrak{p}(L_j)$ holds for all $\mathfrak{p}$, $\mu\geqq q+1$; and so $\mathfrak{a}_{\lambda j}\rho_{\mu j}\omega_\mu\subseteq L_j$. This proves the contention.

By the induction and Lemma 3.4 we can express $\sum_1^{r-1} \mathfrak{a}_{\lambda j}\omega'_{\lambda j} + \sum_r^q \mathfrak{a}_{\lambda j}\omega_\lambda$, $0\leqq j\leqq m$, in a common minimal basis $\zeta_1, \cdots, \zeta_q$. Then $\zeta_1, \cdots, \zeta_q, \omega_{q+1}, \cdots, \omega_n$ is the basis required. We have exhausted all possibilities and the theorem is now proved.

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