1. **Introduction.** Elementary divisor theory over the rational integers is classical; a treatment over an arbitrary algebraic number field can be found in Steinitz [8]. In this paper we are motivated by the following interpretation of Chevalley [4] and Asano [2]: if \( L_1 \) and \( L_2 \) are finite modules in a vector space \( V \) over a Dedekind ring \( \mathfrak{o} \) with quotient field \( F \), then \( L_1 \) and \( L_2 \) have a minimal basis in common. Simple examples show that in general this is not true for more than two modules \( L_1 \) and \( L_2 \), but in Theorem 4.2 we establish conditions under which every module in \( V \) has a minimal basis in common with a given set of modules \( L_0, \ldots, L_m \). The basis structure of modules under a direct sum decomposition of \( V \) is examined in Theorem 4.3.

The results outlined above are proved over any valuation ring \( \mathfrak{o} \); that they can be extended to an arbitrary Dedekind ring is a consequence of Theorem 5.4 which reduces the problem to a local one.

2. **Preliminaries.** Throughout this paper \( \mathfrak{o} \) will denote an integral domain with a unit and with quotient field \( F \); further restrictions will be placed on \( \mathfrak{o} \) in the following paragraphs. The modules \( L \) considered will be torsion-free \((\alpha X = 0 \text{ with } \alpha \in \mathfrak{o}, X \in L, \text{ implies } \alpha = 0 \text{ or } X = 0)\), finitely generated, and of finite rank; hence [2; 4; 6] each \( L \) has a replica in a finite dimensional \( F \)-space \( V \); we will always assume that the modules under consideration are contained in such a vector space \( V \); \( n \) will denote the dimension of this vector space; the subspace of \( V \) that is spanned by \( L \) will be written \( F L \) and we have rank \( L = \dim F \cdot L \leq \dim V = n \).

In addition all modules will have at least one minimal basis in \( V \)—that is, a basis \( \{\xi\} \) for \( V \) and (possibly zero) ideals \( a_i \) in \( F \) such that \( L = a_0 \xi_1 + \cdots + a_n \xi_n \); such a basis is known to exist for finitely generated modules in the following case:

**Theorem 2.1.** Let \( F \) be such that every finitely generated nonzero \( \mathfrak{o} \)-ideal is invertible. If \( L \) is a finitely generated \( \mathfrak{o} \)-module in \( V \) and if \( \xi_1, \ldots, \xi_n \) is any basis for \( V \), then there exist (finitely generated) ideals \( a_i \) and a basis \( \eta_1, \ldots, \eta_n \) for \( V \) such that \( L = \sum a_i \eta_i \).

This is proved in Satz 1 of [2] and the proof found there, though given for Dedekind rings, applies to fields of the above type; it is not necessary to assume that rank \( L = n \). Note that the number of nonzero \( a_i \) is equal to the rank of \( L \). Compare the proofs in [4; 6].
If we say that a result $P(n, m)$, where $n$ and $m$ are non-negative integers, is proved by double induction we mean

\begin{align*}
(2.1) & \quad P(0, m) \quad \text{and} \quad P(n, 0) \quad \text{hold for all } n \text{ and } m, \\
(2.2) & \quad P(n, m - 1) \quad \text{and} \quad P(n - 1, m) \Rightarrow P(n, m).
\end{align*}

One notational matter: we will denote the running indices in a summation by Greek letters, fixed indices by Roman or Gothic letters.

3. **Minimal bases for $L$.** In this section we investigate which bases in $V$ qualify as minimal bases for $L$. The ring $\mathfrak{o}$ is quite general but it is assumed that $L$ is a direct sum of $\mathfrak{o}$-ideals of $F$,

\[(3.1) \quad L = a_1 \xi_1 + \cdots + a_n \xi_n.\]

Let us introduce the $\mathfrak{o}$-module

\[(3.2) \quad M = b_1 \eta_1 + \cdots + b_n \eta_n,\]

the $b_i$ being $\mathfrak{o}$-ideals and $\{\eta\}$ being another basis for $V$ connected with $\{\xi\}$ by the equations

\[(3.3) \quad \eta_i = \sum a_{i\lambda} \xi_\lambda, \quad \xi_j = \sum b_{j\mu} \eta_\mu.\]

**Lemma 3.1.** $L \subseteq M$ if and only if $b_i a_j \subseteq b_i$; $L = M$ if and only if $b_i a_j \subseteq b_i$ and $a_i b_j \subseteq a_i$; for all $i$ and $j$.

**Proof.** It suffices to prove the first part of the lemma. Substituting (3.3) in (3.1) we see that $L \subseteq M$ is equivalent to $\sum a_i (\sum b_{i\lambda} \eta_\lambda) \subseteq \sum b_i \eta_i$; that is, $a_i b_j \subseteq b_i$ for all $i$ and $j$.

**Lemma 3.2.** The basis $\{\eta\}$ is a minimal basis for $L$ if and only if $a_{ij} b_{jk} \subseteq a_i$, for all $i$, $j$, $k$.

**Proof.** The necessity is an immediate consequence of Lemma 3.1: for if $a_{ij} \neq 0$, then $b_j a_i \subseteq b_j \subseteq a_i^{-1}$. To prove the sufficiency we introduce the ideals $b_i = \sum b_{i\lambda} a_\lambda$ and then define $M$ by means of equation (3.2). Then $L = M$ (again by Lemma 3.1) and so $\{\eta\}$ is a minimal basis for $L$.

**Lemma 3.3.** If $L = M$ then $a_{ij} b_j \subseteq a_i$ and $\det \{a_{ij}\} \prod b_i = \prod a_i$; conversely, if these conditions are satisfied and if the $b_i$ are invertible, then $L = M$.

**Proof.** **Necessity.** It follows from Lemma 3.1 and the expansion

$\det \{a_{ij}\} \prod b_i = \prod a_i$. Similarly \(\prod b_i \geq \det \{b_{ij}\} \prod a_i\). The remark $\det \{a_{ij}\} = (\det \{b_{ij}\})^{-1}$ completes the proof of the necessity.
Sufficiency. Let $a_{ij}$ denote the cofactor of $a_{rs}$. Then $a_{ij}' = \sum \pm a_{ir} \cdots a_{nr}$, the first subscript avoiding $r$, the second $s$. Hence $a_{ij}' a_{rs}' \subseteq \prod a_{ik} b_{kj}' = \det \{a_{ij}\}$ and so $b_{rs} a_{ij} \subseteq b_{ij}$. The result follows from Lemma 3.1.

Remark. It follows from the previous lemma that if $L$ and $M$ are isomorphic, then $\prod a_{ik}$ and $\prod b_{kj}$ are in the same ideal class. Compare [6].

Lemma 3.4. Suppose that $L = M$ and let the bases $\{\xi\}$ and $\{\eta\}$ have the property that $F\xi_{m+1} + \cdots + F\xi_n = F\eta_{m+1} + \cdots + F\eta_n$ for some $m$, $0 \leq m \leq n - 1$. Then $a_{m+1}\xi_{m+1} + \cdots + a_n\xi_n = b_{m+1}\eta_{m+1} + \cdots + b_n\eta_n$.

Proof. The given restrictions ensure that all entries $a_{ij}$ and $b_{ij}$ are zero if $i \leq m$ and $j \geq m + 1$. Then $\{a_{ij}\}_{i,j \geq m+1}$ has $\{b_{ij}\}_{i,j \geq m+1}$ as its inverse. But $a_{ij}$ and $b_{ij}$ satisfy the conditions of Lemma 3.1 for all $i,j$ and hence for $i,j \geq m+1$. Q.E.D.

We conclude this section with some remarks on the multiplication of modules by nonzero scalars $t \in F$. (a) $t \cdot L$ is again an $\sigma$-module in $V$. (b) If $\{L_\lambda\}$ is a collection of $\sigma$-modules in $V$ and if $\{\lambda\}$ is a corresponding collection of scalars, then the $L_\lambda$ have a basis $\{\xi\}$ in common if and only if all the $L_\lambda \cdot L_\lambda$ have the same minimal $\sigma$-basis in common. (c) If $\sigma$ is a valuation ring or a Dedekind ring, there are nonzero scalars $t_1, t_2$ such that $t_1 \cdot L \subseteq M$ and $t_2 \cdot L \supseteq M$, provided that $L$ and $M$ are both of rank $n$ (Lemma 3.1).

4. Local theory. The ring $\sigma$ is now taken to be a valuation ring with quotient field $F$: $\sigma$ is an integral domain such that for any $\alpha$ in $F$, either $\alpha \in \sigma$ or $\alpha^{-1} \in \sigma$. Associated with $\sigma$ is a valuation $|\alpha|$, $\alpha \in F$, with values in an ordered multiplicative group $[1,7]$. We denote the maximal prime ideal in $\sigma$ with the letter $p$. Observe that all finitely generated $\sigma$-ideals are principal and hence invertible. All modules will be finitely generated and hence of the form of equation (3.1) with the $a_i$ principal by Theorem 2.1. Hence there is a basis for $L$ in which

$$L = (p_1)\xi_1 + \cdots + (p_n)\xi_n, \quad p_i \in F;$$

putting $\xi' = p_i \xi_i$ we see that we can take $L = \sum a_i\xi'$ if desired. Furthermore, if $L = \sum b_i\eta_i$ is another representation in a minimal basis $\{\eta\}$, then the $b_i$ must all be principal.

Lemma 4.1. Let $L_i = \sum a_i\xi_i = \sum b_i\eta_i$ be modules in $V$, $0 \leq j \leq m$, and let $L_0 = \sum a_i\xi_i = \sum b_i\eta_i$. Then there is a reordering of the $\{\eta\}$ in which $a_{ij} = b_{ij}$ for all $i,j$.

Proof. By induction to $n$. In the notation of (3.3), $\{a_{ij}\}$ must be
unimodular. Now \( \sum a_{1k}b_{1k} = 1 \); therefore there is a \( k \) for which \( |a_{1k}| = 1 = |b_{1k}| = |a_{1k}| \). Interchanging \( \eta_1 \) and \( \eta_k \) we have

\[
|a_{11}| = 1 = |b_{11}|
\]

Then \( a_{11}b_{11} \subseteq a_{1j} \) by Lemma 3.1 and so \( b_{1j} \subseteq a_{1j} \). Similarly \( a_{1j} \subseteq b_{1j} \). Hence

\[
a_{1j} = b_{1j}
\]

Now \( a_{1j}b_{1j} \subseteq a_{1j} \) together with (4.2) and (4.3) implies that \( a_{1k}a_{1j}^{-1}b_{1j} \subseteq b_{1j} \) and therefore \( \sum b_{1j}\eta_k = b_{1j}\eta_1 + \sum b_{1j}(\eta_k - a_{1k}a_{1j}^{-1}\eta_1) \). But \( (\eta_k - a_{1k}a_{1j}^{-1}\eta_1) \subseteq F\xi_2 + \cdots + F\xi_\lambda \) for \( \lambda \geq 2 \); hence \( \sum b_{1j}(\eta_k - a_{1k}a_{1j}^{-1}\eta_1) = \sum a_{1j}\xi_\lambda \), with \( \lambda \geq 2 \), in virtue of Lemma 3.4. Induction completes the proof.

Note. In the case \( m = 1 \) the previous lemma expresses the “uniqueness of the elementary divisors.”

**Theorem 4.2.** Let \( L_j, 0 \leq j \leq m, \) be given rank \( n \) modules in \( V \). Then every rank \( n \) module \( K \) in \( V \) has a minimal basis \( \{x^k\} \) which is also a minimal basis for all the \( L_j \) if and only if either \( tL_j \supseteq L_i \) or \( tL_j \supseteq L_i \) holds for all \( i, j \) and all scalars \( t \in F \).

**Proof.** Sufficiency. By double induction. (2.1) is true by the elementary divisor theorem which holds for valuation rings. We must prove (2.2).

By the induction and (4.1) we can write \( L_0, \ldots, L_{m-1} \) and \( L_m \) in a common minimal basis such that \( L_0 = \sum a_{0\ell}x_\ell, L_j = \sum (p_{\lambda j})x_\lambda, \) the \( (p_{\lambda j}) \) being principal ideals in \( F \). By considering \( tL_j \) if necessary we can make the further assumption that \( L_j \subseteq L_0 \) and \( L_0 \not= \pi^{-1}L_j \supseteq L_0 \) for any \( \pi \in \mathfrak{p}, 0 \leq j \leq m \). Reordering \( L_1, \ldots, L_m \) we obtain \( L_0 \supseteq L_1 \supseteq L_2 \cdots \supseteq L_m \); hence

\[
|p_{\lambda m}| \leq |p_{\lambda, m-1}| \leq \cdots \leq |p_{\lambda 0}| = 1, \quad (p_{\lambda m}) = \mathfrak{p} \text{ or } 0.
\]

Let \( \gamma(\lambda) \) denote the number of \( p \)'s with absolute value 1 in this inequality.

Let \( K \) be of the form \( \sum b_{1\eta}x_\eta \) with \( \xi_1 = \sum b_{1k}\eta_k \). By rearranging \( \{x_\ell\} \) and \( \{\eta_j\} \) we can assume that

\[
|b_{11}| = \max |b_{1\lambda}|, \quad |b_{1\lambda}| = |b_{11}| \Rightarrow \gamma(1) \geq \gamma(\lambda').
\]

Now define \( H_1 = \sum b_{1\lambda}b_{11}^{-1}\eta_\lambda \) and \( \Xi_1 = (\xi_1 - b_{11}b_{11}^{-1}\xi_1) \). Then \( (p_{1j})\xi_1 + (p_{\lambda j})\xi_1 = (p_{1j})\xi_1 + (p_{\lambda j})\Xi_1 \), since \( |b_{11}b_{11}^{-1}| < 1 \) implies \( |p_{1j}b_{11}b_{11}^{-1}| \leq |p_{1j}| \), while if \( |b_{11}b_{11}^{-1}| = 1 \), then \( \gamma(\lambda) \leq \gamma(1) \) implies \( |p_{\lambda j}| \leq |p_{1j}| \). Hence \( L_j = (p_{1j})\xi_1 + L'_j \) where \( L'_j = \sum (p_{\lambda j})\Xi_1 \). Since \( |b_{11}b_{11}^{-1}| \leq 1 \), we must have \( K = \mathfrak{h}H_1 + K' \) where \( K' = \mathfrak{h}\Xi_2 + \cdots + \mathfrak{h}\eta_n \). But \( F\xi_1 = FH_1 \).
and \( \sum_{k} F \Xi_k = F \eta_0 + \cdots + F \eta_0 \) by definition of \( \Xi_k \) and \( \Xi_0 \). In addition \( L_{i+1} \subseteq L_i \) and \( L_{j+1} \supseteq \mathfrak{p} L_j \) by (4.4). Applying the induction again we see that \( L_j', 0 \leq j \leq m \), and \( K' \) have a minimal basis \( \{ \omega \} \) in common. Hence \( \xi_1, \omega_2, \cdots, \omega_0 \) is a minimal basis for \( L_j, 0 \leq j \leq m \), and \( K \). This proves (2.2) and hence establishes the induction.

**Necessity.** Given any pair \( L_i, L_j \) there is a minimal basis \( \{ \xi \} \) such that \( L_i = \sum \delta \xi_k \) and \( L_j = \sum \delta \xi_k \) by the elementary divisor theorem. Hence if the given conditions on the \( L_{ij}, 0 \leq j \leq m \), do not hold there must be a pair, \( L_0 \) and \( L_1 \) say, such that \( L_0 = \sum \delta \xi_k \) and \( tL_1 = (p) \xi_1 + \cdots + (p) \xi_0 \) with \( p \in \mathfrak{p} \) and \( |p| < |\tau| < 1 \), for some \( \tau \in \mathfrak{p} \). By a change of notation let us refer to \( tL_1 \) as \( L_1 \). Now define a new basis \( \{ \eta \} \) for \( V \) by the equations \( \{ \eta \} B = \{ \xi \} \) where

\[
(4.6) \quad \xi_1 = \eta_1, \quad \xi_2 = \pi \eta_1 + p \eta_2, \quad \xi_i = \eta_i, \quad j > 2.
\]

Put \( K = \sum \delta \eta_k \). Contention: \( L_0, L_1 \) and \( K \) do not have a common minimal basis.

Suppose that the contrary is true and let their common minimal basis be

\[
(4.7) \quad \{ \xi \} = \{ \xi \} C = \{ \xi \} B C.
\]

Put \( C = \{ c_{ij} \} \) and \( C^{-1} = \{ d_{ij} \} \). Then the first row and \( k \)th column of \( BC \) has entry \( (c_{1k} + \pi c_{2k}) \) while the \( k \)th row and second column of \( (BC)^{-1} \) has entry \( (d_{k2} - \pi d_{k1})p^{-1} \). Now apply Lemma 3.2 (i) to \( L_0 \): \( c_{1d} d_{k1} \in \mathfrak{p} \) all \( i, j, k \); (ii) to \( L_1 \): \( c_{1d} d_{k1} \in (p) \) for all \( k \); (iii) to \( K \): \( c_{1d} d_{k1} - c_{2d} d_{k2} \in \mathfrak{p} \). Now \( c_{1d} d_{k1} - c_{2d} d_{k2} \in (p) \) by (i) and (ii) and so either \( c_{1d} d_{k1} \in \mathfrak{p} \) or \( c_{2d} d_{k2} \in \mathfrak{p} \) and hence both must be in \( \mathfrak{p} \) by (iv). Hence

\[
1 = \sum c_{1d} d_{k1} \in \mathfrak{p}
\]

and this is a contradiction. Q.E.D.

**Remark.** There is a different way of stating Theorem 4.2—by assuming that \( \tau.0 = V \) but dropping the conditions on the ranks of the \( L_{ij}, 0 \leq j \leq m \), and on \( K \).

**Sufficiency.** It is easy to see that the conditions of the theorem guarantee that all \( L_{ij} \) must be of the same rank and therefore span \( V \). If in addition rank \( K = n \), the proof is as before. If this is not so, write \( K = 0 \cdot \eta_1 + a_2 \eta_2 + \cdots + a_n \eta_n \); instead of (4.5) we can arrange to have

\[
|b_{11}| = \max |b_{1k}|, \quad |b_{1k}| = |b_{11}| \Rightarrow \gamma(1) \geq \gamma(\lambda)
\]

by permuting the \( \{ \xi \} \) only. The rest of the proof is as before.

**Necessity.** If all \( L_{ij}, 0 \leq j \leq m \), are of rank \( n \) the previous proof is valid. If not, we can take \( L_0 = \sum \delta \xi_k \) and \( L_1 = 0 \cdot \xi_1 + \delta \xi_2 + a_2 \xi_2 + \cdots + a_n \xi_n \) by the elementary divisor theorem; put \( K = 0(\xi_1 + \pi^{-1} \xi_0) \) where
Then it is easy to see (e.g. by using Theorem 2.1 and Lemma 3.4) that $L_0$, $L_1$ and $K$ do not have a common minimal basis.

**Theorem 4.3.** If $L_j$, $0 \leq j \leq m$, are modules in $V$ having a minimal basis in common and if each $L_j$ has a decomposition $L_j = L_j' + L_j''$ corresponding to a fixed direct sum decomposition $V = V' + V''$, then the $L_j'$ (resp. $L_j''$) have a common minimal basis in $V'$ (resp. $V''$).

First we prove the following special case.

**Lemma.** The theorem is true if $FL_0 = V$.

**Proof of Lemma.** By double induction. (2.1) is true. We must prove (2.2). By the induction we have

\[
(4.8) \quad L_j' = \sum_{\alpha=1}^{p} (p_{j\alpha})\xi_\alpha \subseteq V', \quad L_j'' = \sum_{\alpha=1}^{n} (p_{j\alpha})\xi_\alpha \subseteq V'',
\]

with $L_0 = \sum_\alpha \xi_\alpha$, that is $p_{0\alpha} = 1$ for all $\alpha$. Let $\{\xi\}$ be a minimal basis common to all the $L_j$, $0 \leq j \leq m$; there is no loss of generality in taking

\[
(4.9) \quad L_0 = \sum_\alpha \xi_\alpha, \quad L_j = \sum (p_{j\alpha})\xi_\alpha, \quad L_m = \sum_\alpha (p_{m\alpha})\xi_\alpha,
\]

If $\xi = \sum c_\alpha \xi_\alpha$, then $\{c_\alpha\}$ must be unimodular and we can permute $\{\xi\}$ as in Lemma 4.1 until $|c_{11}| = 1 = |d_{11}|$, interchange $V'$ and $V''$ if necessary, and then permute $\xi_2, \cdots, \xi_n$ until

\[
(4.10) \quad \sum (p_{\alpha j})\xi_\alpha = L_j = \sum (p_{\alpha j})\xi_\alpha, \quad 0 \leq j \leq m - 1,
\]

\[
(4.11) \quad |c_{11}| = 1, \quad c_{\alpha \beta} p_{j \alpha} \in (p_{j \beta}), \quad 0 \leq j \leq m - 1.
\]

Note that with a corresponding change of notation, (4.8) and (4.9) are preserved.

Now put $Z_1 = \xi_1, \quad Z_\alpha = (\xi_\alpha - c_{\alpha 1} c_{11}^{-1} \xi_1), \quad \Xi_1 = \sum c_{\alpha 1} c_{11}^{-1} \xi_\alpha, \quad \Xi_\alpha = \xi_\alpha$ and then write $\xi$ and $\xi$ instead of $Z$ and $\Xi$. In virtue of (4.11) and these substitutions, (4.8), (4.9) and (4.10) are still valid and

\[
(4.12) \quad \xi_1 = c_{11} \xi_1 + \sum_{\alpha=1}^{n} c_{1\alpha} \xi_\alpha, \quad \xi_\alpha = \sum_{\alpha=2}^{n} c_{\alpha \alpha} \xi_\alpha, \quad i > 1.
\]

It follows from the direct sum decomposition of $L_m$ and (4.9) that

\[
\sum_{\alpha=1}^{n} c_{\alpha 1} \xi_\alpha \subseteq L_m and c_{11} \xi_1 \subseteq L_m. Hence we see from (4.9) and (4.12) that L_m = \sum_\alpha (p_{m\alpha})\xi_\alpha + \sum_\alpha (p_{m\alpha})\xi_\alpha, 0 \leq j \leq m - 1, have $\xi_2, \cdots, \xi_n$ as a common minimal basis in virtue of Lemma 3.4 and (4.12). Contention: there is a direct sum decomposition of $\sum_{\alpha=1}^{n} (p_{m\alpha})\xi_\alpha$ corresponding to $\sum_{\alpha=1}^{n} F_\xi_\alpha \oplus \sum_{\alpha=1}^{n} F_\xi_\alpha$: for $L_m = L'_m + L''_m$;
and \( L'_m = \alpha \xi_1 + \sum_a \alpha_a \eta_a \) with \( \eta_a = \alpha_1 \xi_1 + \cdots + \xi_1 \) in virtue of Theorem 2.1; but \( \alpha_1 \alpha_a \subseteq \theta \) since \( \alpha_m \subseteq \alpha_0 \); hence we can assume that \( \alpha_1 = 0, 2 \leq a \leq q \); then by Lemma 3.4, \( \sum_a (\beta_a) \xi_a = \sum_a \alpha_a \eta_a + \alpha' L''_m \). This proves the contention. Induction completes the proof. Q.E.D.

**Proof of Theorem.** By double induction. By applying the elementary divisor theorem to \( L'_0 \) (resp. \( L'_0'' \)) and \( L'_1 \) (resp. \( L'_1'' \)) we can write

\[
\begin{align*}
L_0 &= \alpha \xi_1 + \cdots + \alpha \xi_s + \cdots + \alpha \xi_t, \\
L_1 &= \beta_1 \xi_r + \cdots + \beta_1 \xi_t + \alpha \xi_{t+1} + \cdots + \alpha \xi_s
\end{align*}
\]

with \( s \leq n \) and \( \{ \xi \} \) such that either \( \xi_1 \in V' \) or \( \xi_1 \in V'' \); there is no loss of generality in assuming that \( \theta_0 \subseteq \sigma, r \leq \lambda \leq t \). On the other hand an easy computation shows that we can assume that equations (4.13) are true if we replace \( \{ \xi \} \) by \( \{ f \} \) where \( \{ f \} \) is a minimal basis common to all \( L_j, 0 \leq j \leq m \). Define \( K \) invariantly as the smallest \( \alpha \)-module containing \( L_0 \) and \( L_1 \). Then

\[
\sum \alpha \xi_n = K = \sum \alpha \lambda.
\]

**Case I.** \( F \cdot K = V \). Then \( K, L_0, L_1, \ldots, L_m \) satisfy the conditions of the lemma; the theorem therefore holds for these modules; hence it holds for \( L_j, 0 \leq j \leq m \).

**Case II.** \( F \cdot K \neq V \). By the induction \( K, L_2, L_3, \ldots, L_m \) have a common basis \( \{ \eta \} \) such that \( K = \sum \alpha \eta \) and either \( \eta_1 \in V' \) or \( \eta_1 \in V'' \) for \( 1 \leq \lambda \leq n \). By means of Lemma 3.4 we see that \( L_0, L_1 \) and \( \sum_1 a_k \eta_k, 2 \leq j \leq m \), must have \( \xi_1, \ldots, \xi_t \) as a common minimal basis and hence have a basis \( \omega_1, \ldots, \omega_n \) with the required property. Then \( \omega_1, \ldots, \omega_n, \eta_{t+1}, \ldots, \eta_n \) is the basis required. This completes the proof of the theorem.

**Lemma 4.4.** Let \( L_j \) be rank \( n \) modules in \( V \) and let \( X \) be any vector in \( L_0 \). Then there is a nonzero \( \alpha \) (independent of \( X \)) such that if \( \{ \xi \} \) is any basis for all \( L_j \) with \( L_0 = \sum a \xi_n \), then so is \( \xi_1, \ldots, \xi_{i-1}, \xi_i + \alpha' X, \xi_{i+1}, \ldots, \xi_n \) where \( |\alpha'| \leq |\alpha| \).

**Proof.** Write \( L_j = \sum a_j \xi_n \) and \( X = \sum a_j \xi_n \) where \( \alpha = \sum \alpha \in \theta; \) choose \( \alpha \in \theta \) such that \( \alpha \in a_{j, k} \alpha_m \) for all \( \lambda, \mu, j \). Then

\[
L_j = a_{j, k} \xi_1 + \cdots + a_{j, k} (\xi_i + \alpha' X) + \cdots + a_{n, k} \xi_n.
\]

5. **Dedekind rings.** In this section we assume that classical ideal theory holds in \( F/\theta \), that is to say the ideals with bounded denominator form a group under multiplication. We denote by \( \theta_p \) the valuation ring corresponding to the prime ideal \( \theta_p \); the same letter \( p \) will be used
for the maximal prime ideal in \( \mathfrak{p} \). The \( \mathfrak{p} \)-ideals in \( F \) are all powers of \( \mathfrak{p} \) and the value group is discrete and real. If \( a = \mathfrak{p}^r \cdots \) is a factorization of the \( \mathfrak{o} \)-ideal \( a \) we define \( \mathfrak{o}(a) \) as the \( \mathfrak{p} \)-ideal \( \mathfrak{p}^r \). All modules will be finitely generated and hence of the form (3.1) with the \( a_i \) of bounded denominator and invertible. Note that if \( \xi_1, \cdots, \xi_n \) is any basis for \( V \), then by Theorem 2.1 there is a minimal basis \( \eta_1, \cdots, \eta_n \) for \( L \) such that

\[
\eta_i = c_{i1}\xi_1 + \cdots + c_{i-1}\xi_{i-1} + \xi_i, \quad c_{ij} \in F.
\]

If \( L = \sum a_\lambda \xi_\lambda \) put \( \mathfrak{o}(L) = \sum \mathfrak{o}(a_\lambda) \xi_\lambda \): that this is well-defined is an immediate consequence of Lemma 3.1. By taking \( \xi_\lambda' = a_\lambda \xi_\lambda \) instead of \( \xi_\lambda \) it is always possible to choose a basis for \( L \) such that \( \mathfrak{o}(L) = \sum \mathfrak{o} \xi_\lambda' \) at a given finite set of primes \( \mathfrak{p} \) provided that rank \( L = n \). By means of Lemma 3.1 and the unique factorization into primes it is easy to prove the following result.

**Lemma 5.1.** \( L = M \) if and only if \( \mathfrak{o}(L) = \mathfrak{o}(M) \) at all \( \mathfrak{p} \). In any case, if \( F \cdot L = F \cdot M \), then \( \mathfrak{o}(L) = \mathfrak{o}(M) \) holds at all but a finite number of primes \( \mathfrak{p} \).

**Lemma 5.2.** Let \( L_i, 0 \leq j \leq m \), be rank \( n \) modules in \( V \) and let \( W \) be an \( r \) dimensional subspace of \( V \). If \( \mathfrak{o}(L_j), 0 \leq j \leq m \), have a common minimal basis \( \{ \eta^p \} \) such that \( \eta^p_1, \cdots, \eta^p_r \in W \) at each \( \mathfrak{p} \) in a finite set of primes \( S \), then there is a basis \( \{ \xi \} \) for \( L_0 \) which is also a basis for all the \( \mathfrak{o}(L_j), \mathfrak{p} \in S \), and such that \( \xi_1, \cdots, \xi_r \in W \).

**Proof.** By induction on \( n \). All bases (whether defined or derived) that appear in the proof will be such that their first \( r \) elements span \( W \). Let us write \( L_0 = \sum a_\lambda \xi_\lambda \) and \( \eta^p_\lambda = \sum a^p_\lambda \xi_\lambda \); we can assume that \( \mathfrak{o}(a_\lambda) = \mathfrak{p} \) and \( \mathfrak{o}(L_0) = \sum \mathfrak{o} \eta^p_\lambda \) at all \( \mathfrak{p} \in S \). Apply the approximation theorem to the entries of the matrices \( \{ a^p_\lambda \} \) in such a way that \( a_{ij} \) approximates to \( a^p_{ij} \) with \( a_{ij} = 0 \) whenever \( a_{ij}^p = 0 \) is true for all \( \mathfrak{p} \in S \); then in virtue of Lemma 4.4 a sufficiently high approximation yields a basis \( \eta_j = \sum a_\lambda \xi_\lambda \) such that

\[
\mathfrak{o}(L_0) = \sum \mathfrak{o} \xi_\lambda, \quad \mathfrak{o}(L_j) = \sum \mathfrak{o} \eta_\lambda, \quad \text{all } \mathfrak{p} \in S.
\]

Now write each \( L_j = \sum a_\lambda \xi_\lambda \) with \( \xi_\lambda \in F \eta_1 + \cdots + F \eta_n \), by (5.1). Then all the modules \( \sum a_\lambda \xi_\lambda \) have \( \eta_1, \cdots, \eta_{n-1} \) as a common minimal basis at each \( \mathfrak{p} \in S \), in virtue of Lemma 3.4; by the induction there is a basis (and we continue to call it \( \{ \xi \} \)) for \( L_0 \) such that \( \eta_1, \cdots, \eta_n \) is a basis for the \( \mathfrak{o}(L_j), \mathfrak{p} \in S \), where \( \eta_i = \xi_i \) for all \( i \leq n - 1 \); we can assume that

\[
\sum \mathfrak{o} \xi_\lambda = \mathfrak{o}(L_0) = \sum \mathfrak{o} \eta_\lambda, \quad \mathfrak{p} \in S.
\]
Suppose that $\eta_n = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$; then $|\alpha_1| \leq 1$ and $|\alpha_n| = 1$ at all $p \in S$; by suitably scaling $\eta_n$ we can make the further assumptions

\[(5.4) \quad \alpha_i \in o, \quad \alpha_i \in a_{00} a_{00}^{-1} \quad \text{for all } i.\]

Let $\alpha, \alpha' \in o$ be such that $|\alpha|$, $|\alpha'|$ are arbitrarily small at $p \in S$, but either $|\alpha| = 1$ or $|\alpha'| = 1$ outside $S$; then g.c.d. $(\alpha, \alpha', \alpha_n) = 0$ and there must exist $\beta, \beta', \gamma \in o$ such that $a\beta + a'\beta' + \gamma \alpha_n = 1$; clearly $\gamma$ must be a unit at all $p \in S$. Define

\[(5.5) \quad \xi_i = \xi_i, \quad i < n, \quad \xi_n = \alpha\beta \xi_n + \alpha'\beta' \xi_n + \gamma \eta_n.\]

Then $\xi_n = \sum_{i}^{-1} \gamma \alpha \xi_n + \xi_n$ and so $\{\xi\}$ is a basis for $L_0$. Since $|\alpha|$, $|\alpha'|$ can be made arbitrarily small at $p \in S$, we have $\{\xi\}$ as a basis for $p(L_j)$, $p \in S$, by Lemma 4.4. Q.E.D.

**Lemma 5.3.** If the assumptions of the previous lemma are satisfied at ALL primes $p$, then there is a common basis $\{\xi\}$ for all $L_j$, $0 \leq j \leq m$, such that $\xi_1, \cdots, \xi_n \in W$.

**Proof.** Let $S$ be a finite set of primes such that $p(L_0) = p(L_j)$ if $p \in S$. Let $\{\xi\}$ be a basis having the properties of Lemma 5.2; then $p(L_0) = \sum p_{\alpha} \xi_\alpha = p(L_j)$ if $p \in S$, and $p(L_j) = \sum p_{\beta} \xi_\beta$ if $p \in S$. Define

\[a_{1j} = \prod_{p \in S} p_{\alpha_j} \prod_{p \in S} p_{\alpha_0}.\]

Then $L_j = \sum a_{1j} \xi_\lambda$ by Lemma 5.1.

**Theorem 5.4.** The modules $L_j$ have a common minimal basis if and only if their local components $p(L_j)$ have a common minimal basis at all primes $p$, $0 \leq j \leq m$.

**Proof.** By double induction. By the elementary divisor theorem

\[L_0 = a_{10} \omega_1 + \cdots + a_{r \alpha} \omega_r + \cdots + a_{q \alpha} \omega_q,\]

\[L_1 = a_{11} \omega_1 + \cdots + a_{q1} \omega_q + \cdots + a_{s1} \omega_s.\]

We can assume that $a_{11} \subseteq a_{10}$, $r \leq \lambda \leq q$. Define $K$ invariantly as the $o$-module generated by $L_0$ and $L_1$; then $K = \sum (a_{10} + a_{11}) \omega_1$.

**Case I.** Rank $L_j = n, 0 \leq j \leq m$. This follows from Lemma 5.3.

**Case II.** Rank $L_0 < n$, rank $L_j = n$, $1 \leq j \leq m$. By Lemma 5.3 we can express $L_j$, $1 \leq j \leq m$, in a common $\{\xi\}$ with $\xi_1, \cdots, \xi_q$ spanning $F \cdot L_0$: $L_j = \sum a_{1j} \xi_\lambda$, $1 \leq j \leq m$. Then $L_0$ and $\sum a_{1j} \xi_\lambda$ have a common basis at all $p$ in virtue of Lemma 3.4 and so by the induction they must have a common basis $\xi_1, \cdots, \xi_q, \xi_{q+1}, \cdots, \xi_n$ in the large. Then $\xi_1, \cdots, \xi_q, \xi_{q+1}, \cdots, \xi_n$ is the basis required.
CASE III. Rank $K = s < n$. Then if $\{\eta^p\}$ is a basis for $\mathfrak{p}(L_j), 0 \leq j \leq m$, it must also be a basis for $\mathfrak{p}(K)$; therefore by the induction $K$, $L_2, \ldots, L_m$ have a common basis $\{\xi\}$; let us suppose that $\xi_1, \ldots, \xi_s$ spans $F \cdot K$; then $L_0, L_1$ and $\sum a_0^j \xi^j$ have a common minimal basis at all $p$ in virtue of Lemma 3.4; hence they have a basis $\xi_1, \ldots, \xi_s$ in the large. Then $\xi_1, \ldots, \xi_s, \xi_{s+1}, \ldots, \xi_n$ is the basis required.

CASE IV. Rank $L_0 < n$, rank $L_1 < n$ and rank $K = s = n$. Then by (5.1) we can express each $L_j$, $2 \leq j \leq m$, in a basis whose successive elements are in $F\omega_n, F\omega_n + F\omega_{n-1},$ etc.; by Lemma 3.4 and the inductive hypothesis we see that we can assume that $L_j = \sum_{i=1}^{r-1} a_i \omega_j + \sum a_i \omega_n$, for $0 \leq j \leq m$.

Contention: we can arrange to have $\omega^p_j \in FL_0$ when $2 \leq j \leq m$. For each $\alpha \in \alpha_j$ we have $a \omega^p_j = \alpha \omega_j + \alpha \omega_{j+1} + \cdots + \cdots \in L_j$; at each $p$, $\mathfrak{p}(L_j)$ has a basis $\{\eta^p\}$ with $\eta^p_0, \ldots, \eta^p_n \in FL_0, \eta^p_n 
\in FL_1$; and $\nu^p_m = \omega_n$ for $\mu \geq r$, by Lemma 3.4, hence

$$a \omega^p_j = \beta_j \eta^p_0 + \cdots + \beta \eta^p_n + \alpha \omega_{j+1} + \cdots + \alpha \omega_n,$$

hence $a \mu \omega_n \in \mathfrak{p}(L_j)$ holds for all $p, \mu \geq q + 1$; and so $a_j \mathfrak{p} \omega_n \subseteq L_j$. This proves the contention.

By the induction and Lemma 3.4 we can express $\sum_{i=1}^{r-1} a_i \omega_j + \sum a_i \omega_n, 0 \leq j \leq m$, in a common minimal basis $\xi_1, \ldots, \xi_s$. Then $\xi_1, \ldots, \xi_q, \omega_{q+1}, \ldots, \omega_n$ is the basis required. We have exhausted all possibilities and the theorem is now proved.

References

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