REGULAR BANACH ALGEBRAS WITH A COUNTABLE SPACE OF MAXIMAL REGULAR IDEALS

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1. Introduction. In a recent paper [6] Wermer showed that, for the Banach algebra $C(D)$ of all complex continuous functions on the unit circle $D$ in the complex plane, the subalgebra $A$ of all functions in $C(D)$ analytic in $|z| < 1$ is a maximal proper closed subalgebra. He also showed that if $L$ is a simple closed curve and $C(L)$ is the Banach algebra of all complex continuous functions on $L$ then $C(L)$ possesses a maximal proper closed subalgebra which separates points of $L$. Rudin [4] has shown that if $X$ is a compact Hausdorff space which contains a subset homeomorphic to the Cantor set then the Banach algebra $C(X)$ contains a maximal proper closed subalgebra which separates points of $X$. In the present note we show that if $X$ is a countable compact Hausdorff space then $C(X)$ contains no such maximal proper closed subalgebra. We obtain this result as a by-product of the study of regular Banach algebras where we investigate a class of subalgebras which, in the case of $C(X)$, are the maximal proper closed subalgebras.

2. Definitions and preliminaries. Let $B$ be a complex commutative Banach algebra with space of maximal regular ideals $\mathfrak{M}$. Let $\pi: x \mapsto x(M)$ be the Gelfand representation of $B$ as a subalgebra of $C(\mathfrak{M})$, the algebra of all complex continuous functions on $\mathfrak{M}$ which vanish at infinity. Where convenient we denote the function $x(M)$ also by $x$. We call a subset $S$ of $B$ a separating family on $\mathfrak{M}$ if for each $M_1, M_2$ in $\mathfrak{M}$, $M_1 \neq M_2$ there exists $x \in S$ such that $x(M_1) \neq x(M_2)$. A subalgebra $A$ of $B$ is called determining if $\pi(A)$ is dense in $\pi(B)$, otherwise $A$ is called nondetermining.

The notions of a maximal proper closed subalgebra and a maximal nondetermining subalgebra are related as follows.

2.1. Lemma. (a) A maximal nondetermining subalgebra $A$ of $B$ is closed in $B$ and $\pi(A)$ is closed in $\pi(B)$.

(b) A maximal proper closed subalgebra $A$ of $B$ which is nondetermining is a maximal nondetermining subalgebra.

(c) A maximal nondetermining subalgebra which is not a separating family on $\mathfrak{M}$ is a maximal proper closed subalgebra and is of the form $\{x \in B | x(M_1) = x(M_2)\}$ where $M_1 \neq M_2$ in $\mathfrak{M}$.

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Proof. (a) We show first that the closure of a nondetermining subalgebra $N$ is also nondetermining. For let $|||x||| = \sup |x(M)|$, $M \in \mathcal{M}$. There exists $\gamma \in B$ and $\varepsilon > 0$ such that $|||\gamma - x||| \geq \varepsilon$ for all $x \in N$. Then it is easy to see that $|||\gamma - z||| \geq \varepsilon$ for all $z \in N$. Thus a maximal nondetermining subalgebra $A$ is closed. If $\pi(A)$ is the closure of $\pi(A)$ in $\pi(B)$ then $\pi^{-1}[\pi(A)^e] = A$ whence $\pi(A)$ is closed in $\pi(B)$.

(b) Let $A_1$ be the subalgebra generated by a maximal proper closed subalgebra $A$ and $x \in A$ where $A$ is nondetermining. Since $A_1$ is dense in $B$ it follows that $\pi(A_1)$ is dense in $\pi(B)$.

(c) From the hypotheses there exist $M_1, M_2 \in \mathcal{M}$, $M_1 \neq M_2$ such that $A \subseteq \{x \in B | x(M_1) = x(M_2)\} = A_1$, say. Since $A_1$ is the null-space of a linear functional on $B$, $A_1$ is a maximal proper closed subalgebra. As $A_1$ is nondetermining, $A = A_1$.

It is clear that any maximal nondetermining subalgebra of $B$ must contain the radical of $B$.

Following the usage in [2] and elsewhere, for a set $\mathfrak{F}$ in $\mathcal{M}$ we define the kernel of $\mathfrak{F}$, $k(\mathfrak{F}) = \bigcap M, M \in \mathfrak{F}$ and for an ideal $I$ in $B$, the hull of $I$ in $\mathcal{M}$ as $h(I) = \{M \in \mathcal{M} | M \supseteq I\}$. $B$ is called regular if the Gelfand topology for $\mathcal{M}$ is the same as the hull-kernel topology for $\mathcal{M}$ (see [2, p. 83] and also [5]). We discuss only regular Banach algebras $B$. Let $\mathfrak{F}$ be a closed set in $\mathcal{M}$ and let $T$ be the natural homomorphism of $B$ onto $B/k(\mathfrak{F})$. As described in [2, p. 76] $T$ defines a mapping $T^*$ of the space $\mathcal{M}$ of maximal regular ideals of $B/k(\mathfrak{F})$ into $\mathcal{M}$ by the rule $x(T^*(N)) = T(x)(N)$ for $x \in B, N \in \mathcal{M}$. Moreover $T^*$ is a homeomorphism and, since $B$ is regular, $T^*(\mathcal{M}) = \mathfrak{F}$. Thus the Gelfand representation of $B/k(\mathfrak{F})$ may be thought of as the restriction of the functions in $\pi(B)$ to the set $\mathfrak{F}$ and we identify $\mathcal{M}$ with $\mathfrak{F}$. For our purposes it is important to note that $B/k(\mathfrak{F})$ is regular [7, p. 164].

3. Regular Banach algebras with $\mathcal{M}$ countable. Throughout this section $B$ denotes a complex commutative regular Banach algebra with space of maximal regular ideals $\mathcal{M}$ where $\mathcal{M}$ is countable. Let $\mathcal{M}^e = \mathcal{M}$ and for each ordinal $\alpha$ let $\mathcal{M}^\alpha$ be the $\alpha$th derived set of $\mathcal{M}$. There is a first ordinal $\beta$ such that $\mathcal{M}^\beta$ is void. For each $\alpha < \beta$ let $T_\alpha$ be the natural homomorphism of $B$ onto $B/k(\mathcal{M}^\alpha)$.

3.1. Lemma. Let $\mathfrak{F}$ be a nonvoid compact subset of $\mathcal{M}$. Then there is a last ordinal $\alpha$ such that $\mathcal{M}^\alpha \cap \mathfrak{F}$ is not void.

Proof. Since $\mathcal{M}^\alpha \cap \mathfrak{F} = \emptyset$ there exists a first ordinal $\mu$ such that $\mathcal{M}^\alpha \cap \mathfrak{F} = \emptyset$. Now $\mu$ cannot be a limit ordinal for otherwise

$$\bigcap_{\tau < \mu} \mathcal{M}^\tau \cap \mathfrak{F} = \emptyset$$
which is impossible since the $M^\alpha \cap \mathfrak{A}$ form a decreasing set of nonvoid compact sets. Then there is an ordinal $\alpha$ such that $\mu = \alpha + 1$.

3.2. Lemma. Let $L$ be a linear manifold in $B$ with the following property. For each $\alpha < \beta$ and each $M \in M^\alpha$ which is an isolated point of $M^\alpha$ there exists $x \in L$ such that the function $x$ restricted to $M^\alpha$ is the characteristic function of $M$. Then the Gelfand representation of $L$ is dense in $C(M)$.

Proof. Let $f(M) \in C(M)$ and $\epsilon > 0$. Let $\mathfrak{U} = \{M \in M : |f(m)| < \epsilon\}$. We show that there exists $g \in L$ such that $|g(M) - f(M)| < \epsilon, M \in M$. We may suppose $f \neq 0$ and that $\epsilon$ is sufficiently small for the complement $\mathfrak{B}$ of $\mathfrak{U}$ to be nonvoid. Since $\mathfrak{B}$ is compact, by Lemma 3.1 there is a last ordinal $\alpha_0$ such that $\mathfrak{B} \cap M^{\alpha_0} \neq \emptyset$. Then $\mathfrak{B} \cap M^{\alpha_0}$ is a finite set, say $M_1, \ldots, M_k$, each point of which is an isolated point of $M^{\alpha_0}$. Let $h_i \in L$, where $h_i$ restricted to $M^{\alpha_0}$ is the characteristic function of $M_i, i = 1, \ldots, k$. A linear combination $w_0$ of the $h_i$ may be chosen so that $w_0(M_i) = f(M_i), i = 1, \ldots, k$. We have $w_0(M) = 0$ for $M \in M^{\alpha_0} \cap \mathfrak{U}$. Thus $|w_0(M) - f(M)| < \epsilon, M \in M^{\alpha_0}$.

If $\alpha_0 = 0$ or if $|w_0(M) - f(M)| < \epsilon$ for all $M \in M$ we have the desired element. Otherwise let $\mathfrak{U}_1 = \{M \in M : |w_0(M) - f(M)| < \epsilon\}$ and let $\mathfrak{B}_1$ be the complement of $\mathfrak{U}_1$. By Lemma 3.1 there is a last ordinal $\alpha_1$ such that $\mathfrak{B}_1 \cap M^{\alpha_1} \neq \emptyset$. Clearly $\alpha_1 < \alpha_0$. By the above procedure we may add a linear combination of elements of $L$ to $w_0$ obtaining $w_1 \in L$ such that $|w_1(M) - f(M)| < \epsilon, M \in M^{\alpha_1}$. If $w_1$ is not the desired element and $\alpha_1 > 0$ we may repeat the procedure. Since a decreasing sequence of ordinals contains only a finite number of terms, at some finite stage we obtain an element $w_n \in L$ such that $|w_n(M) - f(M)| < \epsilon, M \in M$.

3.3. Theorem. (1) The subalgebra $B_0$ of $B$ consisting of all $x \in B$ such that $x$ has compact support has its Gelfand representation dense in $C(M)$.

(2) An ideal $I$ of $B$ is contained in a regular maximal ideal of $B$ if and only if $I$ is nondetermining.

Proof. Consider $M^\alpha$ for $\alpha < \beta$ and $M_0$ an isolated point of $M^\alpha$. The Banach algebra $B/k(M^\alpha)$, being regular, contains an element $x$ whose Gelfand representation is the characteristic function of $M_0$ and $B$ contains an element $y$ where $T_x(y) = x$. Let $\emptyset$ be an open subset of $M$ with compact closure and $M_0 \in \emptyset$. There exists $\bar{z} \in B$ such that $\bar{z}(M_0) = 1$ and $\bar{z}(M) = 0, M \notin \emptyset$. Then for $w = yz$, $w$ restricted to $M^\alpha$ is the characteristic function of $M_0$ and has compact support. The conclusion (1) now follows from Lemma 3.2.
Let \( I \) be an ideal in \( B \). If \( I \) is contained in a maximal regular ideal then clearly \( I \) is nondetermining. Suppose that \( I \) is contained in no maximal regular ideal. Then \( h(I) \) is void. By [2, p. 84, Theorem 24D] we see that \( \pi(B_0) \subseteq \pi(I) \). By (1) of this theorem, \( I \) is determining.

3.4. Lemma. Let \( A \) be a maximal nondetermining subalgebra of \( B \) which is not a maximal regular ideal of \( B \). Suppose that \( M \) is countable. Then either \( \pi(A) \) contains the characteristic functions of all isolated points of \( M \) or \( A \) is not a separating family on \( M \).

Proof. Suppose that \( \pi(A) \) fails to contain the characteristic function \( \delta \) of an isolated point \( M_0 \) of \( M \). Let \( \delta \in A \), \( \pi(\delta) = \delta \). By Lemma 2.1, \( \delta \) is at a positive distance from \( \pi(A) \) in \( C(M) \). Let \( A_1 \) be the algebra generated by \( A \) and \( \delta \) and let \( L \) be the one-dimensional subspace of \( C(M) \) generated by \( \delta \). Let the superscript \( c \) denote closure in \( C(M) \). One easily verifies that \( \pi(A_1) = \pi(A) + L \) is dense in \( \pi(B) \) and that, since \( \pi(A) \) is dense in \( \pi(B) \), we have \( \pi(A)^c + L = \pi(B)^c = C(M) \) by Theorem 3.3. Let \( M_0 \) be the one-point compactification of \( M \) and consider \( C(M_0) \) as a maximal ideal in \( C(M_0) \). There exists a linear functional \( x \) on \( C(M_0) \) which vanishes on \( \pi(A)^c \) and has the property that \( x^*(\delta) = 1 \). Since \( \pi(A) \) is closed in \( \pi(B) \) by Lemma 2.1, \( x^{-1}(0) \cap \pi(B) = \pi(A) \). By the generalized Riesz representation theorem [1] there exists a completely additive regular set function \( \mu \) defined for all Borel sets (and hence for all subsets) of \( M_0 \) such that for \( f \in C(M_0) \),

\[
x^*(f) = \int_{M_0} f(M) d\mu.
\]

Since \( x^*(\delta) = 1 \) we have \( \mu(M_0) = 1 \). It is impossible that \( \mu(M) = 0 \) for all \( M \in M_0 \). \( M \neq M_0 \). For otherwise since there exists \( g \in \pi(A) \) such that \( g(M_0) = 0 \) we have \( x^*(g) = g(M_0) = 0 \) which is impossible.

Let \( M_i \in M_0 \), \( M_i \neq M_0 \), \( i = 1, 2 \) where \( M_1 \neq M_2 \). We show that it is impossible to have \( \mu(M_1) = 0 \) and \( \mu(M_2) = 0 \). Suppose otherwise. Since \( M_0 \) is zero-dimensional there exist open and closed neighborhoods \( \mathcal{V}_i \) of \( M_i \), \( i = 1, 2 \) which are disjoint and contain neither \( M_0 \) nor the point at infinity. It is clear that each \( \mathcal{V}_i \) is an open and closed compact subset of \( M \). By Theorem 3.3, the functions \( \delta, x \in B \), restricted to \( \mathcal{V}_i \) form a dense set in \( C(\mathcal{V}_i) \). Let \( \mu_1 \) be the set function \( \mu \) restricted to subsets of \( \mathcal{V}_1 \). It is clear that \( \mu_1 \) is a completely additive regular set function on \( \mathcal{V}_1 \). Therefore if

\[
\int_{\mathcal{V}_1} f(M) d\mu_1 = 0
\]

for all \( f \in \pi(B) \) we see from [1, p. 39] that (1) holds for all \( f \in C(\mathcal{V}_1) \).
and that \( \mu_1(E) = 0 \) for all Borel sets in \( \mathcal{B}_1 \). This implies that \( \mu(\{ M_1 \}) = 0 \) which is contrary to our assumptions. Hence there exists \( x_1 \in B \) such that

\[
\int_{\mathcal{B}_1} x_1(M) \, d\mu_1 = -1.
\]

By regularity there exists \( y_1 \in B \) such that \( y_1(M) = 0 \), \( M \in \mathcal{B}_1 \) and \( y_1(M_0) = 1 - x_1(M_0) \). Set \( u_1 = x_1 + y_1 \). By a theorem of Silov (see [2, p. 84] and [5, p. 37]) there exists \( v_1 \in B \) such that \( v(M) = 1 \), \( M \in \mathcal{B}_1 \cup \{ M_0 \} \) and \( v(M) = 0 \) elsewhere. Set \( w_1 = u_1 v_1 \). Then \( w_1(M_0) = 1 \), \( w_1(M) = x_1(M) \) for \( M \in \mathcal{B}_1 \) and \( w_1(M) = 0 \) elsewhere. Therefore

\[
x^*(w_1) = w_1(M_0) + \int_{\mathcal{B}_1} w_1(M) \, d\mu_1 = 0.
\]

Hence \( w_1 \in \pi(A) \). In the same way we find \( w_2 \in \pi(A) \) where \( w_2(M_0) = 1 \) and \( w_2(M) = 0 \), \( M \in \mathcal{B}_1 \cup \{ M_0 \} \). But \( \widehat{w}_1 \widehat{w}_2 = 0 \) and \( \widehat{w}_1 \widehat{w}_2 \in \pi(A) \). This contradicts the hypothesis that \( \delta \in \pi(A) \).

Consequently there exists exactly on \( M_1 \in \mathcal{M} \), \( M_1 \neq M_0 \) such that \( \mu(\{ M_1 \}) = \alpha \neq 0 \). For \( f \in \pi(B) \), \( x^*(f) = f(M_0) + \alpha f(M_1) \). Consider \( g \in \pi(A) \). If \( g(M_1) = 0 \) then \( g(M_0) = 0 \). Suppose that \( g(M_1) = -1 \). Then \( g(M_0) - a = 0 \). Since \( g^2 \in \pi(A) \), \( [g(M_0)]^2 + a = 0 \). Then \( a = -1 \). This shows that \( g(M_0) = g(M_1) \) here also. Therefore \( A \) is not a separating family.

3.5. Theorem. The maximal proper closed subalgebras of \( B \) which are nondetermining are the maximal regular ideals of \( B \) and the sets of the form \( \{ x \in B \mid x(M_1) = x(M_2) \}, M_i \in \mathcal{M}, i = 1, 2, M_1 \neq M_2 \} \).

PROOF. In view of Lemma 2.1 it is sufficient to show that a maximal nondetermining subalgebra \( A \) of \( B \) which is not a maximal regular ideal is not a separating family on \( \mathcal{M} \). Suppose otherwise.

We show, by transfinite induction, that \( T_\alpha(A) \), for each \( \alpha < \beta \), is a maximal nondetermining subalgebra of \( B/k(\mathcal{M}^\alpha) \) which is a separating family on \( \mathcal{M}^\alpha \). The conclusion is trivial for \( \alpha = 0 \). Suppose that the assertion is true for all ordinals \( \gamma, \gamma < \alpha < \beta \). We establish first that \( T_\alpha(A) \) is a nondetermining subalgebra of \( B/k(\mathcal{M}^\alpha) \). For suppose otherwise. Let \( f(M) \in \pi(B) \) and \( \epsilon > 0 \). We shall obtain a contradiction by demonstrating that there exists \( g \in A \) such that \( g(M) = f(M) \) \( |g(M) - f(M)| < \epsilon \), \( M \in \mathcal{M} \). There exists \( h \in A \) such that \( |h(M_0) - f(M)| < \epsilon \), \( M \in \mathcal{M}^\alpha \). Let \( \mathcal{U} = \{ M \in \mathcal{M} \mid |h(M) - f(M)| < \epsilon \} \). The complement \( \mathcal{B} \) of \( \mathcal{U} \) is compact. By Lemma 3.1 there is a last ordinal \( \alpha_0 \) such that \( \mathcal{M}^\alpha \cap \mathcal{B} \neq \emptyset \). Clearly \( \alpha_0 < \alpha \). By Lemma 3.4 (applied to the regular Banach algebra \( B/k(\mathcal{M}^\alpha) \)) we see that for each isolated point \( M_0 \) of \( \mathcal{M}^\alpha \), \( A \) contains
an element $x$ such that when $x(M)$ is restricted to $\mathcal{M}^a$, the characteristic function of $M_0$ on $\mathcal{M}^a$ is obtained. Arguing as in Lemma 3.2, we see that there exists $w_t \subseteq A$ such that $|w_t(M) - f(M)| < \epsilon, M \in \mathcal{M}^a$. By repeated use of Lemma 3.4 and by the arguments of Lemma 3.2 we obtain the desired $g \subseteq A$.

We also observe that $T_\alpha(A)$ is a maximal nondetermining subalgebra of $B/k(\mathcal{M})$. For if $Q$ is a nondetermining subalgebra which properly contains $T_\alpha(A)$ then $T_\alpha^{-1}(Q)$ is a nondetermining subalgebra of $B$ which properly contains $A$ (see §2). It is clear that $T_\alpha(A)$ is a separating family on $\mathcal{M}_r$. Thus the induction is complete.

If $\alpha < \beta$ and $M_0$ is an isolated point of $\mathcal{M}_r$, by the above and Lemma 3.4 $A$ possesses an element $x$ such that $x$, restricted to $\mathcal{M}_r$, is the characteristic function of $M_0$. Hence by Lemma 3.2, $A$ is determining. This is a contradiction and the proof is complete.

3.6. Corollary. Let $B$ be a complex commutative $B^*$-algebra with a countable space $\mathcal{M}$ of maximal regular ideals. Then the maximal proper closed subalgebras of $B$ are the maximal regular ideals of $B$ and the sets of the form $\{x \in B | x(M_i) = x(M_j), M_i, M_j \in \mathcal{M}, i = 1, 2, M_1 \neq M_2\}$.

Proof. This follows from Theorem 3.5 since in this case it is clear that the maximal proper closed subalgebras are the maximal nondetermining subalgebras.

Let $G$ be any locally compact abelian group whose character group $\hat{G}$ is countable, e.g. $G$ a generalized toroidal group [3, p. 142]. Let $B$ be the group algebra of $G$. Then $B$ satisfies the hypotheses of Theorems 3.3 and 3.5. Thus the maximal nondetermining subalgebras of $B$ are completely described by the above results.

Bibliography