REGULAR BANACH ALGEBRAS WITH A COUNTABLE SPACE OF MAXIMAL REGULAR IDEALS

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1. Introduction. In a recent paper [6] Wermer showed that, for the Banach algebra \( C(D) \) of all complex continuous functions on the unit circle \( D \) in the complex plane, the subalgebra \( A \) of all functions in \( C(D) \) analytic in \(|z|<1\) is a maximal proper closed subalgebra. He also showed that if \( L \) is a simple closed curve and \( C(L) \) is the Banach algebra of all complex continuous functions on \( L \) then \( C(L) \) possesses a maximal proper closed subalgebra which separates points of \( L \). Rudin [4] has shown that if \( X \) is a compact Hausdorff space which contains a subset homeomorphic to the Cantor set then the Banach algebra \( C(X) \) contains a maximal proper closed subalgebra which separates points of \( X \). In the present note we show that if \( X \) is a countable compact Hausdorff space then \( C(X) \) contains no such maximal proper closed subalgebra. We obtain this result as a by-product of the study of regular Banach algebras where we investigate a class of subalgebras which, in the case of \( C(X) \), are the maximal proper closed subalgebras.

2. Definitions and preliminaries. Let \( B \) be a complex commutative Banach algebra with space of maximal regular ideals \( M \). Let \( \pi: x \mapsto x(M) \) be the Gelfand representation of \( B \) as a subalgebra of \( C(M) \), the algebra of all complex continuous functions on \( M \) which vanish at infinity. Where convenient we denote the function \( x(M) \) also by \( x \). We call a subset \( S \) of \( B \) a separating family on \( M \) if for each \( M_1, M_2 \) in \( M \), \( M_1 \neq M_2 \) there exists \( x \in S \) such that \( x(M_1) \neq x(M_2) \). A subalgebra \( A \) of \( B \) is called determining if \( \pi(A) \) is dense in \( \pi(B) \), otherwise \( A \) is called nondetermining.

The notions of a maximal proper closed subalgebra and a maximal nondetermining subalgebra are related as follows.

2.1. Lemma. (a) A maximal nondetermining subalgebra \( A \) of \( B \) is closed in \( B \) and \( \pi(A) \) is closed in \( \pi(B) \).

(b) A maximal proper closed subalgebra \( A \) of \( B \) which is nondetermining is a maximal nondetermining subalgebra.

(c) A maximal nondetermining subalgebra which is not a separating family on \( M \) is a maximal proper closed subalgebra and is of the form \( \{ x \in B : x(M_1) = x(M_2) \} \) where \( M_1 \neq M_2 \) in \( M \).

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PROOF. (a) We show first that the closure of a nondetermining subalgebra $N$ is also nondetermining. For let $|||x||| = \sup |x(M)|$, $M \in \mathcal{M}$. There exists $y \in \mathcal{B}$ and $\epsilon > 0$ such that $|||y-x||| \geq \epsilon$ for all $x \in N$. Then it is easy to see that $|||y-z||| \geq \epsilon$ for all $z \in N$. Thus a maximal nondetermining subalgebra $A$ is closed. If $\pi(A)^{\circ}$ is the closure of $\pi(A)$ in $\pi(\mathcal{B})$ then $\pi^{-1}[\pi(A)^{\circ}] = A$ whence $\pi(A)$ is closed in $\pi(\mathcal{B})$.

(b) Let $A_{1}$ be the subalgebra generated by a maximal proper closed subalgebra $A$ and $x \in A$ where $A$ is nondetermining. Since $A_{1}$ is dense in $\mathcal{B}$ it follows that $\pi(A_{1})$ is dense in $\pi(\mathcal{B})$.

(c) From the hypotheses there exist $M_{1}, M_{2} \in \mathcal{M}, M_{1} \neq M_{2}$ such that $A \subset \{x \in \mathcal{B} | x(M_{1}) = x(M_{2})\} = A_{1}$, say. Since $A_{1}$ is the null-space of a linear functional on $\mathcal{B}$, $A_{1}$ is a maximal proper closed subalgebra. As $A_{1}$ is nondetermining, $A = A_{1}$.

It is clear that any maximal nondetermining subalgebra of $\mathcal{B}$ must contain the radical of $\mathcal{B}$.

Following the usage in [2] and elsewhere, for a set $\mathfrak{g}$ in $\mathcal{M}$ we define the kernel of $\mathfrak{g}$, $k(\mathfrak{g}) = \bigcap M, M \in \mathfrak{g}$ and for an ideal $I$ in $\mathcal{B}$, the hull of $I$ in $\mathcal{M}$ as $h(I) = \{M \in \mathcal{M} | M \supset I\}$. $\mathcal{B}$ is called regular if the Gelfand topology for $\mathcal{M}$ is the same as the hull-kernel topology for $\mathcal{M}$ (see [2, p. 83] and also [5]). We discuss only regular Banach algebras $\mathcal{B}$. Let $\mathfrak{g}$ be a closed set in $\mathcal{M}$ and let $T$ be the natural homomorphism of $\mathcal{B}$ onto $\mathcal{B}/k(\mathfrak{g})$. As described in [2, p. 76] $T$ defines a mapping $T^{*}$ of the space $\mathcal{N}$ of maximal regular ideals of $\mathcal{B}/k(\mathfrak{g})$ into $\mathcal{M}$ by the rule $x(T^{*}(N)) = T(x)(N)$ for $x \in \mathcal{B}, N \in \mathcal{M}$. Moreover $T^{*}$ is a homeomorphism and, since $\mathcal{B}$ is regular, $T^{*}(\mathcal{N}) = \mathfrak{g}$. Thus the Gelfand representation of $\mathcal{B}/k(\mathfrak{g})$ may be thought of as the restriction of the functions in $\pi(\mathcal{B})$ to the set $\mathfrak{g}$ and we identify $\mathcal{N}$ with $\mathfrak{g}$. For our purposes it is important to note that $\mathcal{B}/k(\mathfrak{g})$ is regular [7, p. 164].

3. Regular Banach algebras with $\mathcal{M}$ countable. Throughout this section $\mathcal{B}$ denotes a complex commutative regular Banach algebra with space of maximal regular ideals $\mathcal{M}$ where $\mathcal{M}$ is countable. Let $\mathcal{M}^{\alpha} = \mathcal{M}$ and for each ordinal $\alpha$ let $\mathcal{M}^{\alpha}$ be the $\alpha$th derived set of $\mathcal{M}$. There is a first ordinal $\beta$ such that $\mathcal{M}^{\beta} = \mathcal{M}$ and for each $\alpha < \beta$ let $T_{\alpha}$ be the natural homomorphism of $\mathcal{B}$ onto $\mathcal{B}/k(\mathcal{M}^{\alpha})$.

3.1. Lemma. Let $\mathfrak{g}$ be a nonvoid compact subset of $\mathcal{M}$. Then there is a last ordinal $\alpha$ such that $\mathcal{M}^{\alpha} \cap \mathfrak{g}$ is not void.

PROOF. Since $\mathcal{M}^{\beta} \cap \mathfrak{g} = \emptyset$ there exists a first ordinal $\mu$ such that $\mathcal{M}^{\nu} \cap \mathfrak{g} = \emptyset$. Now $\mu$ cannot be a limit ordinal for otherwise

$$\bigcap_{\gamma < \mu} \mathcal{M}^{\gamma} \cap \mathfrak{g} = \emptyset$$
which is impossible since the $\mathfrak{M}^\alpha \cap \mathfrak{N}$ form a decreasing set of nonvoid compact sets. Then there is an ordinal $\alpha$ such that $\mu = \alpha + 1$.

3.2. Lemma. Let $L$ be a linear manifold in $B$ with the following property. For each $\alpha < \beta$ and each $M \in \mathfrak{M}^\alpha$ which is an isolated point of $\mathfrak{M}^\alpha$ there exists $x \in L$ such that the function $x$ restricted to $\mathfrak{M}^\alpha$ is the characteristic function of $M$. Then the Gelfand representation of $L$ is dense in $C(\mathfrak{M})$.

Proof. Let $f(M) \in C(\mathfrak{M})$ and $\epsilon > 0$. Let $\mathcal{U} = \{ M \in \mathfrak{M} \mid |f(M)| < \epsilon \}$.

We show that there exists $g \in L$ such that $|g(M) - f(M)| < \epsilon$, $M \in \mathfrak{M}$. We may suppose $f \neq 0$ and that $\epsilon$ is sufficiently small for the complement $\mathfrak{N}$ of $\mathfrak{U}$ to be nonvoid. Since $\mathfrak{N}$ is compact, by Lemma 3.1 there is a last ordinal $\alpha_0$ such that $\mathfrak{N} \cap \mathfrak{M}^{\alpha_0} \neq \emptyset$. Then $\mathfrak{N} \cap \mathfrak{M}^{\alpha_0}$ is a finite set, say $M_1, \ldots, M_k$, each point of which is an isolated point of $\mathfrak{M}^{\alpha_0}$. Let $h_i \in L$, where $h_i$ restricted to $\mathfrak{M}^{\alpha_i}$ is the characteristic function of $M_i$, $i = 1, \ldots, k$. A linear combination $w_0$ of the $h_i$ may be chosen so that $w_0(M_i) = f(M_i)$, $i = 1, \ldots, k$. We have $w_0(M) = 0$ for $M \in \mathfrak{M}^{\alpha_0} \cap \mathcal{U}$. Thus $|w_0(M) - f(M)| < \epsilon$, $M \in \mathfrak{M}^{\alpha_0}$.

If $\alpha_0 = 0$ or if $|w_0(M) - f(M)| < \epsilon$ for all $M \in \mathfrak{M}$ we have the desired element. Otherwise let $\mathcal{U}_1 = \{ M \in \mathfrak{M} \mid |w_0(M) - f(M)| < \epsilon \}$ and let $\mathfrak{N}_1$ be the complement of $\mathcal{U}_1$. By Lemma 3.1 there is a last ordinal $\alpha_1$ such that $\mathfrak{N}_1 \cap \mathfrak{M}^{\alpha_1} \neq \emptyset$. Clearly $\alpha_1 < \alpha_0$. By the above procedure we may add a linear combination of elements of $L$ to $w_0$ obtaining $w_1 \in L$ such that $|w_1(M) - f(M)| < \epsilon$, $M \in \mathfrak{M}^{\alpha_1}$. If $w_1$ is not the desired element and $\alpha_1 > 0$ we may repeat the procedure. Since a decreasing sequence of ordinals contains only a finite number of terms, at some finite stage we obtain an element $w_n \in L$ such that $|w_n(M) - f(M)| < \epsilon$, $M \in \mathfrak{M}$.

3.3. Theorem. (1) The subalgebra $B_0$ of $B$ consisting of all $x \in B$ such that $x$ has compact support has its Gelfand representation dense in $C(\mathfrak{M})$.

(2) An ideal $I$ of $B$ is contained in a regular maximal ideal of $B$ if and only if $I$ is nondetermining.

Proof. Consider $\mathfrak{M}^\alpha$ for $\alpha < \beta$ and $M_0$ an isolated point of $\mathfrak{M}^\alpha$. The Banach algebra $B/k(\mathfrak{M}^\alpha)$, being regular, contains an element $x$ whose Gelfand representation is the characteristic function of $M_0$ and $B$ contains an element $y$ where $T_\alpha(y) = x$. Let $\emptyset$ be an open subset of $\mathfrak{M}$ with compact closure and $M_0 \in \emptyset$. There exists $z \in B$ such that $z(M_0) = 1$ and $z(M) = 0$, $M \notin \emptyset$. Then for $w = yz$, $\tilde{w}$ restricted to $\mathfrak{M}^\alpha$ is the characteristic function of $M_0$ and has compact support. The conclusion (1) now follows from Lemma 3.2.
Let $I$ be an ideal in $B$. If $I$ is contained in a maximal regular ideal then clearly $I$ is nondetermining. Suppose that $I$ is contained in no maximal regular ideal. Then $h(I)$ is void. By [2, p. 84, Theorem 24D] we see that $\pi(B_0) \subseteq \pi(I)$. By (1) of this theorem, $I$ is determining.

3.4. Lemma. Let $A$ be a maximal nondetermining subalgebra of $B$ which is not a maximal regular ideal of $B$. Suppose that $\mathcal{M}$ is countable. Then either $\pi(A)$ contains the characteristic functions of all isolated points of $\mathcal{M}$ or $A$ is not a separating family on $\mathcal{M}$.

Proof. Suppose that $\pi(A)$ fails to contain the characteristic function $\delta$ of an isolated point $M_0$ of $\mathcal{M}$. Let $\delta \in A$, $\pi(\delta) = \delta$. By Lemma 2.1, $\delta$ is at a positive distance from $\pi(A)$ in $C(\mathcal{M})$. Let $A_1$ be the algebra generated by $A$ and $\delta$, and let $L$ be the one-dimensional subspace of $C(\mathcal{M})$ generated by $\delta$. Let the superscript $c$ denote closure in $C(\mathcal{M})$. One easily verifies that $\pi(A_1) = \pi(A) + L$ is dense in $\pi(B)$ and that, since $\pi(A) + L$ is closed in $\pi(B)^c$, we have $\pi(A)^c + L = (\pi(B))^c = C(\mathcal{M})$ by Theorem 3.3. Let $\mathcal{M}_0$ be the one-point compactification of $\mathcal{M}$ and consider $C(\mathcal{M}_0)$ as a maximal ideal in $C(\mathcal{M}_0)$. There exists a linear functional $x$ on $C(\mathcal{M}_0)$ which vanishes on $\pi(A)^c$ and has the property that $x^*(\delta) = 1$. Since $\pi(A)$ is closed in $\pi(B)$ by Lemma 2.1, $x^*: \pi(B)^c = \pi(A)$. By the generalized Riesz representation theorem [1] there exists a completely additive regular set function $\mu$ defined for all Borel sets (and hence for all subsets) of $\mathcal{M}_0$ such that for $f \in C(\mathcal{M}_0)$,

$$x^*(f) = \int_{\mathcal{M}_0} f(M) d\mu.$$

Since $x^*(\delta) = 1$ we have $\mu(\{M_0\}) = 1$. It is impossible that $\mu(\{M\}) = 0$ for all $M \in \mathcal{M}_0$, $M \neq M_0$. For otherwise since there exists $g \in \pi(A)$ such that $g(M_0) \neq 0$ we have $x^*(g) = g(M_0) \neq 0$ which is impossible.

Let $M_i \in \mathcal{M}$, $M_i \neq M_0$, $i = 1, 2$ where $M_1 \neq M_2$. We show that it is impossible to have $\mu(\{M_1\}) \neq 0$ and $\mu(\{M_2\}) \neq 0$. Suppose otherwise. Since $\mathcal{M}_0$ is zero-dimensional there exist open and closed neighborhoods $\mathcal{B}_i$ of $M_i$, $i = 1, 2$ which are disjoint and contain neither $M_0$ nor the point at infinity. It is clear that each $\mathcal{B}_i$ is an open and closed compact subset of $\mathcal{M}$. By Theorem 3.3, the functions $\check{x}$, $x \in B$, restricted to $\mathcal{B}_i$ form a dense set in $C(\mathcal{B}_i)$. Let $\mu_i$ be the set function $\mu$ restricted to subsets of $\mathcal{B}_i$. It is clear that $\mu_i$ is a completely additive regular set function on $\mathcal{B}_i$. Therefore if

$$\int_{\mathcal{B}_i} f(M) d\mu_i = 0$$

for all $f \in \pi(B)$ we see from [1, p. 39] that (1) holds for all $f \in C(\mathcal{B}_i)$.
and that \( \mu_1(E) = 0 \) for all Borel sets in \( \mathcal{B}_1 \). This implies that \( \mu(\{M_1\}) = 0 \) which is contrary to our assumptions. Hence there exists \( x_1 \in B \) such that

\[
\int_{\mathcal{B}_1} x_1(M) d\mu_1 = -1.
\]

By regularity there exists \( y_1 \in B \) such that \( y_1(M) = 0 \), \( M \in \mathcal{B}_1 \) and \( y_1(M_0) = 1 - x_1(M_0) \). Set \( u_1 = x_1 + y_1 \). By a theorem of Silov (see [2, p. 84] and [5, p. 37]) there exists \( v_1 \in B \) such that \( v(M) = 1 \), \( M \in \mathcal{B}_1 \cup \{M_0\} \) and \( v(M) = 0 \) elsewhere. Set \( w_1 = u_1 v_1 \). Then \( w_1(M_0) = 1 \), \( w_1(M) = x_1(M) \) for \( M \in \mathcal{B}_1 \) and \( w_1(M) = 0 \) elsewhere. Therefore

\[
x_*(w_1) = w_1(M_0) + \int_{\mathcal{B}_1} w_1(M) d\mu_1 = 0.
\]

Hence \( w_1 \in \pi(A) \). In the same way we find \( w_2 \in \pi(A) \) where \( w_2(M_0) = 1 \) and \( w_2(M) = 0 \), \( M \in \mathcal{B}_1 \) and \( w_2(M_0) = 0 \) elsewhere. But \( w_1 w_2 = \delta \) and \( \delta \in \pi(A) \). This contradicts the hypothesis that \( \delta \in \pi(A) \).

Consequently there exists exactly on \( M_1 \in \mathcal{M} \), \( M_1 \neq M_0 \) such that \( \mu(\{M_1\}) = a \neq 0 \). For \( f \in \pi(B) \), \( x^*(f) = f(M_0) + af(M_1) \). Consider \( g \in \pi(A) \). If \( g(M_1) = 0 \) then \( g(M_0) = 0 \). Suppose that \( g(M_1) = -1 \). Then \( g(M_0) = a = 0 \). Since \( g^2 \in \pi(A) \), \( [g(M_0)]^2 + a = 0 \). Then \( a = -1 \). This shows that \( g(M_0) = g(M_1) \) here also. Therefore \( A \) is not a separating family.

3.5. Theorem. The maximal proper closed subalgebras of \( B \) which are nondetermining are the maximal regular ideals of \( B \) and the sets of the form \( \{x \in B | x(M_1) = x(M_2) \}, M_i \in \mathcal{M}, i = 1, 2, M_1 \neq M_2 \} \).

Proof. In view of Lemma 2.1 it is sufficient to show that a maximal nondetermining subalgebra \( A \) of \( B \) which is not a maximal regular ideal is not a separating family on \( \mathcal{M} \). Suppose otherwise.

We show, by transfinite induction, that \( T_\alpha(A) \), for each \( \alpha < \beta \), is a maximal nondetermining subalgebra of \( B/k(\mathcal{M}^\alpha) \) which is a separating family on \( \mathcal{M}^\alpha \). The conclusion is trivial for \( \alpha = 0 \). Suppose that the assertion is true for all ordinals \( \gamma, \gamma < \alpha < \beta \). We establish first that \( T_\alpha(A) \) is a nondetermining subalgebra of \( B/k(\mathcal{M}^\alpha) \). For suppose otherwise. Let \( f(M) \in \pi(B) \) and \( \epsilon > 0 \). We shall obtain a contradiction by demonstrating that there exists \( g \in A \) such that \( |g(M) - f(M)| < \epsilon, M \in \mathcal{M} \). There exists \( h \in A \) such that \( |h(M) - f(M)| < \epsilon, M \in \mathcal{M}^\alpha \). Let \( U = \{M \in M | |h(M) - f(M)| < \epsilon \} \). The complement \( \mathcal{B} \) of \( U \) is compact. By Lemma 3.1 there is a last ordinal \( \alpha_0 \) such that \( \mathcal{M}_0 \cap \mathcal{B} \neq \emptyset \). Clearly \( \alpha_0 < \alpha \). By Lemma 3.4 (applied to the regular Banach algebra \( B/k(\mathcal{M}^\alpha) \)) we see that for each isolated point \( M_0 \) of \( \mathcal{M}^\alpha \), \( A \) contains
an element \( x \) such that when \( x(M) \) is restricted to \( \mathcal{M}^a \) the characteristic function of \( M_0 \) on \( \mathcal{M}^a \) is obtained. Arguing as in Lemma 3.2, we see that there exists \( w_1 \in A \) such that \( | w_1(M) - f(M) | < \epsilon \), \( M \in \mathcal{M}^a \). By repeated use of Lemma 3.4 and by the arguments of Lemma 3.2 we obtain the desired \( g \in A \).

We also observe that \( T_\alpha(A) \) is a maximal nondetermining subalgebra of \( B/k(\mathcal{M}^a) \). For if \( Q \) is a nondetermining subalgebra which properly contains \( T_\alpha(A) \) then \( T_\alpha^{-1}(Q) \) is a nondetermining subalgebra of \( B \) which properly contains \( A \) (see §2). It is clear that \( T_\alpha(A) \) is a separating family on \( \mathcal{M}^a \). Thus the induction is complete.

If \( \alpha < \beta \) and \( M_0 \) is an isolated point of \( \mathcal{M}^a \), by the above and Lemma 3.4 \( A \) possesses an element \( x \) such that \( x \), restricted to \( \mathcal{M}^a \), is the characteristic function of \( M_0 \). Hence by Lemma 3.2, \( A \) is determining. This is a contradiction and the proof is complete.

3.6. Corollary. Let \( B \) be a complex commutative \( B^* \)-algebra with a countable space \( \mathcal{M} \) of maximal regular ideals. Then the maximal proper closed subalgebras of \( B \) are the maximal regular ideals of \( B \) and the sets of the form \( \{ x \in B \mid x(M_1) = x(M_2), M_i \in \mathcal{M}, i = 1, 2, M_1 \neq M_2 \} \).

Proof. This follows from Theorem 3.5 since in this case it is clear that the maximal proper closed subalgebras are the maximal nondetermining subalgebras.

Let \( G \) be any locally compact abelian group whose character group \( \hat{G} \) is countable, e.g. \( G \) a generalized toroidal group [3, p. 142]. Let \( B \) be the group algebra of \( G \). Then \( B \) satisfies the hypotheses of Theorems 3.3 and 3.5. Thus the maximal nondetermining subalgebras of \( B \) are completely described by the above results.

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