REGULAR BANACH ALGEBRAS WITH A COUNTABLE SPACE OF MAXIMAL REGULAR IDEALS

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1. **Introduction.** In a recent paper [6] Wermer showed that, for the Banach algebra $C(D)$ of all complex continuous functions on the unit circle D in the complex plane, the subalgebra A of all functions in $C(D)$ analytic in $|z|<1$ is a maximal proper closed subalgebra. He also showed that if L is a simple closed curve and $C(L)$ is the Banach algebra of all complex continuous functions on L then $C(L)$ possesses a maximal proper closed subalgebra which separates points of L. Rudin [4] has shown that if X is a compact Hausdorff space which contains a subset homeomorphic to the Cantor set then the Banach algebra $C(X)$ contains a maximal proper closed subalgebra which separates points of X. In the present note we show that if X is a countable compact Hausdorff space then $C(X)$ contains no such maximal proper closed subalgebra. We obtain this result as a by-product of the study of regular Banach algebras where we investigate a class of subalgebras which, in the case of $C(X)$, are the maximal proper closed subalgebras.

2. **Definitions and preliminaries.** Let B be a complex commutative Banach algebra with space of maximal regular ideals $\mathfrak{M}$. Let π: $x \rightarrow x(M)$ be the Gelfand representation of B as a subalgebra of $C(\mathfrak{M})$, the algebra of all complex continuous functions on $\mathfrak{M}$ which vanish at infinity. Where convenient we denote the function $x(M)$ also by $\hat{x}$. We call a subset S of B a *separating family* on $\mathfrak{M}$ if for each M_1, M_2 in $\mathfrak{M}$, $M_1 \neq M_2$ there exists $x \in S$ such that $x(M_1) \neq x(M_2)$. A subalgebra A of B is called *determining* if $\pi(A)$ is dense in $\pi(B)$, otherwise A is called *nondetermining*.

The notions of a maximal proper closed subalgebra and a maximal nondetermining subalgebra are related as follows.

2.1. LEMMA. (a) *A maximal nondetermining subalgebra A of B is closed in B and $\pi(A)$ is closed in $\pi(B)$.*

(b) *A maximal proper closed subalgebra A of B which is nondetermining is a maximal nondetermining subalgebra.*

(c) *A maximal nondetermining subalgebra which is not a separating family on $\mathfrak{M}$ is a maximal proper closed subalgebra and is of the form $\{x \in B \mid x(M_1) = x(M_2)\}$ where $M_1 \neq M_2$ in $\mathfrak{M}$.*

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PROOF. (a) We show first that the closure of a nondetermining subalgebra N is also nondetermining. For let $|||x|||=\sup |x(M)|$, $M\in\mathfrak{M}$. There exists $y\in B$ and $\epsilon>0$ such that $|||y-x|||\geqq\epsilon$ for all $x\in N$. Then it is easy to see that $|||y-z|||\geqq\epsilon$ for all $z\in\overline{N}$. Thus a maximal nondetermining subalgebra A is closed. If $\pi(A)^c$ is the closure of $\pi(A)$ in $\pi(B)$ then $\pi^{-1}[\pi(A)^c]=A$ whence $\pi(A)$ is closed in $\pi(B)$.

(b) Let A_1 be the subalgebra generated by a maximal proper closed subalgebra A and $x\notin A$ where A is nondetermining. Since A_1 is dense in B it follows that $\pi(A_1)$ is dense in $\pi(B)$.

(c) From the hypotheses there exist M_1, $M_2\in\mathfrak{M}$, $M_1\neq M_2$ such that $A\subset\{x\in B\,|\,x(M_1)=x(M_2)\}=A_1$, say. Since A_1 is the null-space of a linear functional on B, A_1 is a maximal proper closed subalgebra. As A_1 is nondetermining, $A=A_1$.

It is clear that any maximal nondetermining subalgebra of B must contain the radical of B.

Following the usage in [2] and elsewhere, for a set $\mathfrak{F}$ in $\mathfrak{M}$ we define the *kernel* of $\mathfrak{F}$, $k(\mathfrak{F})=\bigcap M$, $M\in\mathfrak{F}$ and for an ideal I in B, the hull of I in $\mathfrak{M}$ as $h(I)=\{M\in\mathfrak{M}\,|\,M\supset I\}$. B is called *regular* if the Gelfand topology for $\mathfrak{M}$ is the same as the hull-kernel topology for $\mathfrak{M}$ (see [**2**, p. 83] and also [**5**]). We discuss only regular Banach algebras B. Let $\mathfrak{F}$ be a closed set in $\mathfrak{M}$ and let T be the natural homomorphism of B onto $B/k(\mathfrak{F})$. As described in [**2**, p. 76] T defines a mapping T^* of the space $\mathfrak{N}$ of maximal regular ideals of $B/k(\mathfrak{F})$ into $\mathfrak{M}$ by the rule $x(T^*(N))=T(x)(N)$ for $x\in B$, $N\in\mathfrak{N}$. Moreover T^* is a homeomorphism and, since B is regular, $T^*(\mathfrak{N})=\mathfrak{F}$. Thus the Gelfand representation of $B/k(\mathfrak{F})$ may be thought of as the restriction of the functions in $\pi(B)$ to the set $\mathfrak{F}$ and we identify $\mathfrak{N}$ with $\mathfrak{F}$. For our purposes it is important to note that $B/k(\mathfrak{F})$ is regular [**7**, p. 164].

3. **Regular Banach algebras with $\mathfrak{M}$ countable.** Throughout this section B denotes a complex commutative regular Banach algebra with space of maximal regular ideals $\mathfrak{M}$ where $\mathfrak{M}$ is countable. Let $\mathfrak{M}^0=\mathfrak{M}$ and for each ordinal α let $\mathfrak{M}^\alpha$ be the αth derived set of $\mathfrak{M}$. There is a first ordinal β such that $\mathfrak{M}^\beta$ is void. For each $\alpha<\beta$ let T_α be the natural homomorphism of B onto $B/k(\mathfrak{M}^\alpha)$.

3.1. LEMMA. *Let $\mathfrak{F}$ be a nonvoid compact subset of $\mathfrak{M}$. Then there is a last ordinal α such that $\mathfrak{M}^\alpha\cap\mathfrak{F}$ is not void.*

PROOF. Since $\mathfrak{M}^\beta\cap\mathfrak{F}=\emptyset$ there exists a first ordinal μ such that $\mathfrak{M}^\mu\cap\mathfrak{F}=\emptyset$. Now μ cannot be a limit ordinal for otherwise

$$\bigcap_{\gamma<\mu}\mathfrak{M}^\gamma\cap\mathfrak{F}=\emptyset$$

which is impossible since the $\mathfrak{M}^{\gamma}\cap\mathfrak{F}$ form a decreasing set of nonvoid compact sets. Then there is an ordinal α such that $\mu=\alpha+1$.

3.2. LEMMA. *Let L be a linear manifold in B with the following property. For each $\alpha<\beta$ and each $M\in\mathfrak{M}^{\alpha}$ which is an isolated point of $\mathfrak{M}^{\alpha}$ there exists $x\in L$ such that the function x restricted to $\mathfrak{M}^{\alpha}$ is the characteristic function of M. Then the Gelfand representation of L is dense in $C(\mathfrak{M})$.*

PROOF. Let $f(M)\in C(\mathfrak{M})$ and $\epsilon>0$. Let $\mathfrak{U}=\{M\in\mathfrak{M}\mid |f(m)|<\epsilon\}$. We show that there exists $g\in L$ such that $|g(M)-f(M)|<\epsilon$, $M\in\mathfrak{M}$. We may suppose $f\neq 0$ and that ϵ is sufficiently small for the complement $\mathfrak{V}$ of $\mathfrak{U}$ to be nonvoid. Since $\mathfrak{V}$ is compact, by Lemma 3.1 there is a last ordinal α_0 such that $\mathfrak{V}\cap\mathfrak{M}^{\alpha_0}\neq\emptyset$. Then $\mathfrak{V}\cap\mathfrak{M}^{\alpha_0}$ is a finite set, say $M_1, \cdots, M_k$, each point of which is an isolated point of $\mathfrak{M}^{\alpha_0}$. Let $h_i\in L$, where h_i restricted to $\mathfrak{M}^{\alpha_0}$ is the characteristic function of M_i, $i=1, \cdots, k$. A linear combination w_0 of the h_i may be chosen so that $w_0(M_i)=f(M_i)$, $i=1, \cdots, k$. We have $w_0(M)=0$ for $M\in\mathfrak{M}^{\alpha_0}\cap\mathfrak{U}$. Thus $|w_0(M)-f(M)|<\epsilon$, $M\in\mathfrak{M}^{\alpha_0}$.

If $\alpha_0=0$ or if $|w_0(M)-f(M)|<\epsilon$ for all $M\in\mathfrak{M}$ we have the desired element. Otherwise let $\mathfrak{U}_1=\{M\in\mathfrak{M}\mid |w_0(M)-f(M)|<\epsilon\}$ and let $\mathfrak{V}_1$ be the complement of $\mathfrak{U}_1$. By Lemma 3.1 there is a last ordinal α_1 such that $\mathfrak{V}_1\cap\mathfrak{M}^{\alpha_1}\neq\emptyset$. Clearly $\alpha_1<\alpha_0$. By the above procedure we may add a linear combination of elements of L to w_0 obtaining $w_1\in L$ such that $|w_1(M)-f(M)|<\epsilon$, $M\in\mathfrak{M}^{\alpha_1}$. If w_1 is not the desired element and $\alpha_1>0$ we may repeat the procedure. Since a decreasing sequence of ordinals contains only a finite number of terms, at some finite stage we obtain an element $w_n\in L$ such that $|w_n(M)-f(M)|<\epsilon$, $M\in\mathfrak{M}$.

3.3. THEOREM. (1) *The subalgebra B_0 of B consisting of all $x\in B$ such that $\hat{x}$ has compact support has its Gelfand representation dense in $C(\mathfrak{M})$.*

(2) *An ideal I of B is contained in a regular maximal ideal of B if and only if I is nondetermining.*

PROOF. Consider $\mathfrak{M}^{\alpha}$ for $\alpha<\beta$ and M_0 an isolated point of $\mathfrak{M}^{\alpha}$. The Banach algebra $B/k(\mathfrak{M}^{\alpha})$, being regular, contains an element x whose Gelfand representation is the characteristic function of M_0 and B contains an element y where $T_{\alpha}(y)=x$. Let $\mathfrak{G}$ be an open subset of $\mathfrak{M}$ with compact closure and $M_0\in\mathfrak{G}$. There exists $z\in B$ such that $z(M_0)=1$ and $z(M)=0$, $M\notin\mathfrak{G}$. Then for $w=yz$, $\hat{w}$ restricted to $\mathfrak{M}^{\alpha}$ is the characteristic function of M_0 and has compact support. The conclusion (1) now follows from Lemma 3.2.

Let I be an ideal in B. If I is contained in a maximal regular ideal then clearly I is nondetermining. Suppose that I is contained in no maximal regular ideal. Then $h(I)$ is void. By [2, p. 84, Theorem 24D] we see that $\pi(B_0)\subset\pi(I)$. By (1) of this theorem, I is determining.

3.4. LEMMA. *Let A be a maximal nondetermining subalgebra of B which is not a maximal regular ideal of B. Suppose that $\mathfrak{M}$ is countable. Then either $\pi(A)$ contains the characteristic functions of all isolated points of $\mathfrak{M}$ or A is not a separating family on $\mathfrak{M}$.*

PROOF. Suppose that $\pi(A)$ fails to contain the characteristic function δ of an isolated point M_0 of $\mathfrak{M}$. Let $\delta_1\in A$, $\pi(\delta_1)=\delta$. By Lemma 2.1, δ is at a positive distance from $\pi(A)$ in $C(\mathfrak{M})$. Let A_1 be the algebra generated by A and δ_1 and let L be the one-dimensional subspace of $C(\mathfrak{M})$ generated by δ. Let the superscript c denote closure in $C(\mathfrak{M})$. One easily verifies that $\pi(A_1)=\pi(A)+L$ is dense in $\pi(B)$ and that, since $\pi(A)^c+L$ is closed in $\pi(B)^c$, we have $\pi(A)^c+L=\pi(B)^c=C(\mathfrak{M})$ by Theorem 3.3. Let $\mathfrak{M}_0$ be the one-point compactification of $\mathfrak{M}$ and consider $C(\mathfrak{M})$ as a maximal ideal in $C(\mathfrak{M}_0)$. There exists a linear functional x on $C(\mathfrak{M}_0)$ which vanishes on $\pi(A)^c$ and has the property that $x^*(\delta)=1$. Since $\pi(A)$ is closed in $\pi(B)$ by Lemma 2.1, $x^{*-1}(0)\cap\pi(B)=\pi(A)$. By the generalized Riesz representation theorem [1] there exists a completely additive regular set function μ defined for all Borel sets (and hence for all subsets) of $\mathfrak{M}_0$ such that for $f\in C(\mathfrak{M}_0)$,

$$x^*(f) = \int_{\mathfrak{M}_0} f(M)d\mu.$$

Since $x^*(\delta)=1$ we have $\mu(\{M_0\})=1$. It is impossible that $\mu(\{M\})=0$ for all $M\in\mathfrak{M}_0$, $M\neq M_0$. For otherwise since there exists $g\in\pi(A)$ such that $g(M_0)\neq 0$ we have $x^*(g)=g(M_0)\neq 0$ which is impossible.

Let $M_i\in\mathfrak{M}$, $M_i\neq M_0$, $i=1, 2$ where $M_1\neq M_2$. We show that it is impossible to have $\mu(\{M_1\})\neq 0$ and $\mu(\{M_2\})\neq 0$. Suppose otherwise. Since $\mathfrak{M}_0$ is zero-dimensional there exist open and closed neighborhoods $\mathfrak{B}_i$ of M_i, $i=1, 2$ which are disjoint and contain neither M_0 nor the point at infinity. It is clear that each $\mathfrak{B}_i$ is an open and closed compact subset of $\mathfrak{M}$. By Theorem 3.3, the functions $\hat{x}$, $x\in B$, restricted to $\mathfrak{B}_1$ form a dense set in $C(\mathfrak{B}_1)$. Let μ_1 be the set function μ restricted to subsets of $\mathfrak{B}_1$. It is clear that μ_1 is a completely additive regular set function on $\mathfrak{B}_1$. Therefore if

$$\int_{\mathfrak{B}_1} f(M)d\mu_1 = 0 \tag{1}$$

for all $f\in\pi(B)$ we see from [1, p. 39] that (1) holds for all $f\in C(\mathfrak{B}_1)$

and that $\mu_1(E)=0$ for all Borel sets in $\mathfrak{B}_1$. This implies that $\mu(\{M_1\})=0$ which is contrary to our assumptions. Hence there exists $x_1 \in B$ such that

$$\int_{\mathfrak{B}_1} x_1(M)d\mu_1 = -1.$$

By regularity there exists $y_1 \in B$ such that $y_1(M)=0$, $M\in\mathfrak{B}_1$ and $y_1(M_0)=1-x_1(M_0)$. Set $u_1=x_1+y_1$. By a theorem of Šilov (see [**2**, p. 84] and [**5**, p. 37]) there exists $v_1 \in B$ such that $v(M)=1$, $M\in\mathfrak{B}_1 \cup\{M_0\}$ and $v_1(M)=0$ elsewhere. Set $w_1=u_1v_1$. Then $w_1(M_0)=1$, $w_1(M)=x_1(M)$ for $M\in\mathfrak{B}_1$ and $w_1(M)=0$ elsewhere. Therefore

$$x^*(w_1) = w_1(M_0) + \int_{\mathfrak{B}_1} w_1(M)d\mu_1 = 0.$$

Hence $w_1\in\pi(A)$. In the same way we find $w_2\in\pi(A)$ where $w_2(M_0)=1$ and $w_2(M)=0$, $M\notin\mathfrak{B}_2\cup\{M_0\}$. But $\widehat{w}_1\widehat{w}_2=\delta$ and $\widehat{w}_1\widehat{w}_2\in\pi(A)$. This contradicts the hypothesis that $\delta\notin\pi(A)$.

Consequently there exists exactly on $M_1\in\mathfrak{M}$, $M_1\neq M_0$ such that $\mu(\{M_1\})=a\neq 0$. For $f\in\pi(B)$, $x^*(f)=f(M_0)+af(M_1)$. Consider $g\in\pi(A)$. If $g(M_1)=0$ then $g(M_0)=0$. Suppose that $g(M_1)=-1$. Then $g(M_0)-a=0$. Since $g^2\in\pi(A)$, $[g(M_0)]^2+a=0$. Then $a=-1$. This shows that $g(M_0)=g(M_1)$ here also. Therefore A is not a separating family.

3.5. Theorem. *The maximal proper closed subalgebras of B which are nondetermining are the maximal regular ideals of B and the sets of the form* $\{x\in B \mid x(M_1)=x(M_2),\ M_i\in\mathfrak{M},\ i=1, 2,\ M_1\neq M_2\}$.

Proof. In view of Lemma 2.1 it is sufficient to show that a maximal nondetermining subalgebra A of B which is not a maximal regular ideal is not a separating family on $\mathfrak{M}$. Suppose otherwise.

We show, by transfinite induction, that $T_\alpha(A)$, for each $\alpha<\beta$, is a maximal nondetermining subalgebra of $B/k(\mathfrak{M}^\alpha)$ which is a separating family on $\mathfrak{M}^\alpha$. The conclusion is trivial for $\alpha=0$. Suppose that the assertion is true for all ordinals γ, $\gamma<\alpha<\beta$. We establish first that $T_\alpha(A)$ is a nondetermining subalgebra of $B/k(\mathfrak{M}^\alpha)$. For suppose otherwise. Let $f(M)\in\pi(B)$ and $\epsilon>0$. We shall obtain a contradiction by demonstrating that there exists $g\in A$ such that $|g(M)-f(M)|<\epsilon$, $M\in\mathfrak{M}$. There exists $h\in A$ such that $|h(M)-f(M)|<\epsilon$, $M\in\mathfrak{M}^\alpha$. Let $\mathfrak{U}=\{M\in M \mid |h(M)-f(M)|<\epsilon\}$. The complement $\mathfrak{B}$ of $\mathfrak{U}$ is compact. By Lemma 3.1 there is a last ordinal α_0 such that $\mathfrak{M}^{\alpha_0}\cap\mathfrak{B}\neq\emptyset$. Clearly $\alpha_0<\alpha$. By Lemma 3.4 (applied to the regular Banach algebra $B/k(\mathfrak{M}^{\alpha_0})$) we see that for each isolated point M_0 of $\mathfrak{M}^{\alpha_0}$, A contains an element x such that when $x(M)$ is restricted to $\mathfrak{M}^{\alpha_0}$ the characteristic function of M_0 on $\mathfrak{M}^{\alpha_0}$ is obtained. Arguing as in Lemma 3.2, we see that there exists $w_1 \in A$ such that $|w_1(M) - f(M)| < \epsilon$, $M \in \mathfrak{M}^{\alpha_0}$. By repeated use of Lemma 3.4 and by the arguments of Lemma 3.2 we obtain the desired $g \in A$.

We also observe that $T_\alpha(A)$ is a maximal nondetermining subalgebra of $B/k(\mathfrak{M}^\alpha)$. For if Q is a nondetermining subalgebra which properly contains $T_\alpha(A)$ then $T_\alpha^{-1}(Q)$ is a nondetermining subalgebra of B which properly contains A (see §2). It is clear that $T_\alpha(A)$ is a separating family on $\mathfrak{M}^\alpha$. Thus the induction is complete.

If $\alpha < \beta$ and M_0 is an isolated point of $\mathfrak{M}^\alpha$, by the above and Lemma 3.4 A possesses an element x such that $\hat{x}$, restricted to $\mathfrak{M}^\alpha$, is the characteristic function of M_0. Hence by Lemma 3.2, A is determining. This is a contradiction and the proof is complete.

3.6. Corollary. *Let B be a complex commutative B^*-algebra with a countable space $\mathfrak{M}$ of maximal regular ideals. Then the maximal proper closed subalgebras of B are the maximal regular ideals of B and the sets of the form* $\{x \in B \mid x(M_1) = x(M_2),\ M_i \in \mathfrak{M},\ i = 1, 2,\ M_1 \neq M_2\}$.

Proof. This follows from Theorem 3.5 since in this case it is clear that the maximal proper closed subalgebras are the maximal nondetermining subalgebras.

Let G be any locally compact abelian group whose character group $\hat{G}$ is countable, e.g. G a generalized toroidal group [3, p. 142]. Let B be the group algebra of G. Then B satisfies the hypotheses of Theorems 3.3 and 3.5. Thus the maximal nondetermining subalgebras of B are completely described by the above results.

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