ON THE REPRESENTATION OF INDEFINITE INTEGRALS CONTAINING BESSEL FUNCTIONS BY SIMPLE NEUMANN SERIES[1]

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Indefinite integrals containing Bessel functions and their representation as simple Neumann series or alternatively in terms of Lommel's functions of two variables have been noted in the literature in connection with physical problems [**1**; **2**; **3**]. It is observed here that by a simple generalization of a result noted by Watson [**4**, p. 23, footnote] expressing a generating function for Bessel's function as an indefinite integral, all of the previously noted examples may be obtained as particular cases of a more general result, which is then applied to the evaluation of an integral arising in connection with noise theory.

If we define

$$S_\nu(f, g) = \sum_{m=0}^{\infty} f^{m+\nu} J_{m+\nu}(g) \tag{1}$$

where f and g are functions of a parameter t, then differentiation of (1), using the recurrence relations

$$\begin{aligned} J_{\nu-1} - J_{\nu+1} &= 2J_\nu', \\ J_{\nu-1} + J_{\nu+1} &= (2\nu/t)J_\nu \end{aligned} \tag{2}$$

(in which the argument of J_ν is t) gives

$$S_\nu' - \frac{1}{2}(fg - g/f)'S_\nu = \frac{1}{2}(fg)'f^{\nu-1}J_{\nu-1}(g) + \frac{1}{2}(g/f)'f^\nu J_\nu(g). \tag{3}$$

Here, and in succeeding equations primes denote differentiation with respect to t. Solution of (3) gives

$$\begin{aligned} &S_\nu \exp\left[-\frac{1}{2}(fg - g/f)\right] \\ &= \frac{1}{2}\int \exp\left[-\frac{1}{2}(fg - g/f)\right]\left[(fg)'f^{\nu-1}J_{\nu-1}(g) + (g/f)'f^\nu J_\nu(g)\right]dt \end{aligned} \tag{4}$$

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in which it is clear that in (1) and (4) J_ν may be replaced by any other Bessel function satisfying (2). Since it will be useful for further examples we may write the result, similar to (4), obtained by defining

$$T_\nu(f, g) = \sum_{m=0}^{\infty} f^{m+\nu} I_{m+\nu}(g) \tag{5}$$

where I_ν satisfies the recurrence relations

$$\begin{aligned} I_{\nu-1} - I_{\nu+1} &= (2\nu/t) I_\nu, \\ I_{\nu-1} + I_{\nu+1} &= 2I_\nu'. \end{aligned} \tag{6}$$

Following the procedure used to obtain (4) we have

$$\begin{aligned} &T_\nu \exp\left[-\frac{1}{2}(fg + g/f)\right] \\ &= \frac{1}{2}\int \exp\left[-\frac{1}{2}(fg + g/f)\right]\left[(fg)' f^{\nu-1} I_{\nu-1}(g) - (g/f)' f^\nu I_\nu(g)\right] dt. \end{aligned} \tag{7}$$

The most useful particular cases of (4) and (7) may be obtained by assuming f and g to satisfy

$$\alpha fg + \beta g/f = \gamma \tag{8}$$

where α, β, γ are independent of t.

The main point of this paper is expressed in Equations (4) and (7) and the expression of the integrals appearing in these equations in terms of S_ν and T_ν when f and g satisfy (8). The specific forms which these expressions take, namely Equations (17′), (20′), (31′) and (32′) which relate to (4) and Equations (24′), (25′), (35′) and (36′) which relate to (7), will differ somewhat depending on whether or not $\gamma = 0$. Equations (17′), (20′), (24′) and (25′) are for $\gamma \neq 0$ and Equations (31′), (32′), (35′) and (36′) are for $\gamma = 0$.

Let us first assume $\gamma \neq 0$. Then from (8) either f and g are each dependent on t or both are independent of t. In the latter case (3) is trivial. In the former, using (8), (4) may be written in terms of f alone, in the form

$$\beta F_{\nu-1} - \alpha F_\nu = Q_\nu \tag{9}$$

where

$$\begin{aligned} F_\nu = \gamma \int &\exp\left[\frac{1}{2}\gamma(1 - f^2)/(\alpha f^2 + \beta)\right] \\ &\cdot (\alpha f^2 + \beta)^{-2} f^{\nu+1} J_\nu(\gamma f/(\alpha f^2 + \beta))\, df \end{aligned} \tag{10}$$

and

(11) $$Q_\nu = S_\nu \exp\left[\frac{1}{2}\gamma(1-f^2)/(\alpha f^2+\beta)\right].$$

Note that if either $\alpha=0$ or $\beta=0$ then from (9), (10), (11) the integral in (10) may be expressed directly in terms of S_ν. If $\alpha\beta\neq 0$ then substituting

(12) $$F_\nu = \delta^{-\nu} G_\nu, \qquad \delta = \alpha/\beta$$

in (9) we have

(13) $$G_{\nu-1} - G_\nu = \alpha^{-1}\delta^\nu Q_\nu$$

from which

(14) $$G_{\nu-1} - G_{\nu+n} = \alpha^{-1}\sum_{j=0}^{n}\delta^{\nu+j}Q_{\nu+j}.$$

Hence if

(15) $$\lim_{n\to\infty} G_{\nu+n} = 0,$$

then substituting (1) and (11) in (14) and reversing the order of summation we have

(16) $$G_{\nu-1} = \alpha^{-1}\exp\left[\frac{1}{2}\gamma(1-f^2)/(\alpha f^2+\beta)\right]\sum_{m=0}^{\infty} f^{m+\nu}J_{m+\nu}\sum_{j=0}^{m}\delta^{\nu+j}.$$

In Equations (16), (18)–(20) and (25) the argument of $J_{m+\nu}$ and I_ν is understood to be $\gamma f/(\alpha f^2+\beta)$. Thus if $\alpha\neq\beta$, the sum over j in (16) is $\delta^\nu(1-\delta^{m+1})/(1-\delta)$ and

(17) $$G_{\nu-1}=(\beta-\alpha)^{-1}\exp\left[\frac{1}{2}\gamma(1-f^2)/(\alpha f^2+\beta)\right]\left[\delta^{\nu-1}S_\nu(f,g)-S_\nu(\delta f,g)\right].$$

Collecting the expressions comprising (17) we have, assuming (15),

(17′) $$\begin{aligned}\gamma\int \exp\left[\frac{1}{2}\gamma(1-f^2)/(\alpha f^2+\beta)\right](\alpha f^2+\beta)^{-2}f^\nu J_{\nu-1}(\gamma f/(\alpha f^2+\beta))df \\ = (\beta-\alpha)^{-1}\exp\left[\frac{1}{2}\gamma(1-f^2)/(\alpha f^2+\beta)\right] \\ \cdot\left[\sum_{m=0}^{\infty} f^{m+\nu}J_{m+\nu}(\gamma f/(\alpha f^2+\beta))\right. \\ \left. - (\alpha/\beta)^{1-\nu}\sum_{m=0}^{\infty}(\alpha f/\beta)^{m+\nu}J_{m+\nu}(\gamma f/(\alpha f^2+\beta))\right].\end{aligned}$$

However, if $\alpha=\beta$ then

$$(18)\quad \sum_{m=0}^{\infty} f^{m+\nu}J_{m+\nu}\sum_{j=0}^{m}\delta^{\nu+j} = \sum_{m=0}^{\infty} f^{m+\nu}(m+\nu)J_{m+\nu} - (\nu-1)\sum_{m=0}^{\infty} f^{m+\nu}J_{m+\nu}.$$

Substituting (2) in the first series on the right hand side of (18) and making use of (1) and (8), with $\alpha=\beta$, we have

$$(19)\quad \sum_{m=0}^{\infty} f^{m+\nu}(m+\nu)J_{m+\nu} = \frac{1}{2}\gamma S_\nu/\alpha + \frac{1}{2}g(f^\nu J_{\nu-1} - f^{\nu-1}J_\nu).$$

From (16), (18) and (19) we have

$$(20)\quad \begin{aligned} G_{\nu-1} &= \alpha^{-1}\exp\left[\frac{1}{2}\gamma(1-f^2)/(\alpha+\alpha f^2)\right] \\ &\quad\cdot\left[(1-\nu+\frac{1}{2}\gamma/\alpha)S_\nu + \frac{1}{2}g(f^\nu J_{\nu-1} - f^{\nu-1}J_\nu)\right]. \end{aligned}$$

Collecting the expressions in (20), setting $\gamma/2\alpha=c$ and assuming (15) with $\alpha=\beta$, we have

$$(20')\quad \begin{aligned} 2c\int &\exp\,[c(1-f^2)/(1+f^2)](1+f^2)^{-2}f^\nu J_{\nu-1}(2cf/(1+f^2))df \\ &= \exp\,[c(1-f^2)/(1+f^2)]\Big\{(1-\nu+c)\sum_{m=0}^{\infty} f^{m+\nu}J_{m+\nu}(2cf/(1+f^2)) \\ &\quad + [cf/(1+f^2)][f^\nu J_{\nu-1}(2cf/(1+f^2)) - f^{\nu-1}J_\nu(2cf/(1+f^2))]\Big\}. \end{aligned}$$

The equivalent formulas with I_ν replacing J_ν may be obtained from (7) and (8) in similar fashion and are, with

$$(21)\quad \begin{aligned} D_\nu &= \gamma\int \exp\left[-\frac{1}{2}\gamma(f^2+1)/(\alpha f^2+\beta)\right] \\ &\quad\cdot(\alpha f^2+\beta)^{-2}f^{\nu+1}I_\nu(\gamma f/(\alpha f^2+\beta))df; \end{aligned}$$

$$(22)\quad D_\nu = (-\delta)^{-\nu}E_\nu$$

and assuming

$$(23)\quad \lim_{n\to\infty} E_{\nu+n} = 0,$$

$$E_{\nu-1} = (\alpha+\beta)^{-1}\exp\left[-\frac{1}{2}\gamma(f^2+1)/(\alpha f^2+\beta)\right] \cdot [(-\delta)^{\nu-1}T_\nu(f,g) - T_\nu(-\delta f, g)] \tag{24}$$

if $\alpha+\beta\neq 0$.

Collecting the expressions in (24) we have, assuming (23),

$$\begin{aligned}\gamma\int \exp\left[-\frac{1}{2}\gamma(1+f^2)/(\alpha f^2+\beta)\right](\alpha f^2+\beta)^{-2}f^\nu I_{\nu-1}(\gamma f/(\alpha f^2+\beta))df \\ = (\alpha+\beta)^{-1}\exp\left[-\frac{1}{2}\gamma(1+f^2)/(\alpha f^2+\beta)\right] \\ \cdot\left[\sum_{m=0}^{\infty} f^{m+\nu}I_{m+\nu}(\gamma f/(\alpha f^2+\beta)) \right. \\ \left. - (-\alpha/\beta)^{1-\nu}\sum_{m=0}^{\infty}(-\alpha f/\beta)^{m+\nu}I_{m+\nu}(\gamma f/(\alpha f^2+\beta))\right].\end{aligned} \tag{24'}$$

However, if $\alpha=-\beta$ then

$$E_{\nu-1} = \beta^{-1}\exp\left[\frac{1}{2}\gamma(f^2+1)/(\beta f^2-\beta)\right] \cdot\left[\left(1-\nu-\frac{1}{2}\gamma/\beta\right)T_\nu + \frac{1}{2}g(f^\nu I_{\nu-1}+f^{\nu-1}I_\nu)\right]. \tag{25}$$

Collecting the expressions in (25), setting $\gamma/2\beta=c$ and assuming (23) with $\alpha=-\beta$, we have

$$\begin{aligned}2c\int \exp[c(1+f^2)/(f^2-1)](1-f^2)^{-2}f^\nu I_{\nu-1}(2cf/(1-f^2))df \\ = \exp[c(1+f^2)/(f^2-1)]\left\{(1-\nu-c)\sum_{m=0}^{\infty}f^{m+\nu}I_{m+\nu}(2cf/(1-f^2)) \right. \\ \left. + [cf/(1-f^2)][f^\nu I_{\nu-1}(2cf/(1-f^2)) + f^{\nu-1}I_\nu(2cf/(1-f^2))]\right\}.\end{aligned} \tag{25'}$$

Finally, if $\gamma=0$, then from (8)

$$f^2 = -\beta/\alpha \tag{26}$$

since $g=0$ gives trivial results. Thus

$$f - 1/f \equiv 2\kappa = \text{const.} \tag{27}$$

and from (4)

$$S_\nu e^{-\kappa g} = \frac{1}{2}\int e^{-\kappa g}[f^\nu J_{\nu-1}(g) + f^{\nu-1}J_\nu(g)]dg. \tag{28}$$

Using (26), we may write (28) in the form of (9) where now

$$F_\nu = \left(\frac{1}{2} f^{\nu+1}/\beta\right)\int e^{-\kappa g}J_\nu(g)dg \tag{29}$$

and

$$Q_\nu = S_\nu e^{-\kappa g}. \tag{30}$$

Thus, defining G_ν in terms of F_ν by (12) we now have, following the procedure used in deriving (17),

$$G_{\gamma-1} = (\beta - \alpha)^{-1}e^{-\kappa g}/[\delta^{\nu-1}S_\nu(f, g) - S_\nu(\delta f, g)] \tag{31}$$

if $\alpha \neq \beta$.

Collecting the terms in (31) and assuming (15) with G_ν and F_ν defined by (12) and (29) respectively, we have

$$\begin{aligned}&\int e^{-\kappa g}J_{\nu-1}(g)dg \\ &= 2(\beta - \alpha)^{-1}e^{-\kappa g}\left[\beta \sum_{m=0}^{\infty} f^m J_{m+\nu}(g) - \alpha \sum_{m=0}^{\infty} (\alpha f/\beta)^m J_{m+\nu}(g)\right]\end{aligned} \tag{31$'$}$$

where f and κ are given in terms of α and β by (26) and (27). From (20), if $\alpha = \beta$,

$$G_{\nu-1} = \alpha^{-1}e^{-\kappa g}\left[(1 - \nu)S_\nu + \frac{1}{2} g(f^\nu J_{\nu-1}(g) - f^{\nu-1}J_\nu(g))\right]. \tag{32}$$

Collecting the terms in (32), noting that $\alpha = \beta$ and hence from (26) and (27) $\kappa = f = \pm i$, and assuming (15) with $G_\nu = F_\nu$ defined by (29), we have

$$\begin{aligned}&\int e^{\mp ig}J_{\nu-1}(g)dg \\ &= e^{\mp ig}\left[2(1 - \nu) \sum_{m=0}^{\infty} (\pm i)^m J_{m+\nu}(g) + g(J_{\nu-1}(g) \pm iJ_\nu(g))\right].\end{aligned} \tag{32$'$}$$

Similarly, if $\gamma = 0$,

$$f + 1/f \equiv 2\lambda = \text{const.} \tag{33}$$

and from (7), with

(34) $$D_\nu = \left(\frac{1}{2} f^{\nu+1}/\beta\right) \int e^{-\lambda g} I_\nu(g) dg$$

and E_ν defined in terms of D_ν as in (22), we have, following the procedure used in deriving (24),

(35) $$E_{\nu-1} = (\alpha + \beta)^{-1} e^{-\lambda g}[(-\delta)^{\nu-1} T_\nu(f, g) - T_\nu(-\delta f, g)]$$

if $\alpha+\beta \neq 0$.

Collecting the terms in (35) and assuming (23) with D_ν and E_ν defined by (22) and (34) respectively, we have

(35′) $$\int e^{-\lambda g} I_{\nu-1}(g) dg = 2(\alpha + \beta)^{-1} e^{-\lambda g}\left[\beta \sum_{m=0}^{\infty} f^m I_{m+\nu}(g) + \alpha \sum_{m=0}^{\infty} (-\alpha f/\beta)^m I_{m+\nu}(g)\right]$$

where f and λ are given in terms of α and β by (26) and (33). From (25), if $\alpha+\beta = 0$,

(36) $$E_{\nu-1} = \beta^{-1} e^{-\lambda g}\left[(1 - \nu) T_\nu + \frac{1}{2} g(f^\nu I_{\nu-1}(g) + f^{\nu-1} I_\nu(g))\right].$$

Collecting the terms in (36), noting that $\alpha+\beta = 0$ and hence from (26) and (27) $\lambda = f = \pm 1$, and assuming (23) with $D_\nu = E_\nu$ defined by (34), we have

(36′) $$\int e^{\mp g} I_{\nu-1}(g) dg = e^{\mp g}\left[2(1 - \nu) \sum_{m=0}^{\infty} (\pm 1)^m I_{m+\nu}(g) + g(I_{\nu-1}(g) \pm I_\nu(g))\right].$$

We may now consider a few particular examples.

(1) If in (12), (29) and (31) we set $f = 1$, then $\kappa = 0$, $\beta = -\alpha$ and

$$\int J_{\nu-1}(g) dg = S_\nu(1, g) + (-1)^\nu S_\nu(-1, g) = 2 \sum_{m=0}^{\infty} J_{\nu+2m}(g)$$

since it may be shown directly that (15) is satisfied. Thus if Re $(\nu) > 0$,

$$\int_0^x J_{\nu-1}(t) dt = 2 \sum_{m=0}^{\infty} J_{\nu+2m}(x).$$[2]

[2] Reference [4, p. 545, Equation (9)].

(2) If in (7) we set $f=t/a$, $g=at$, then

$$T_\nu \exp\left[-\frac{1}{2}(t^2+a^2)\right] = \int \exp\left[-\frac{1}{2}(t^2+a^2)\right] t(t/a)^{\nu-1} I_{\nu-1}(at)\,dt$$

so that if Re $(\nu)>0$,

$$a^{\nu-1}\exp\left[-\frac{1}{2}(x^2+a^2)\right]\sum_{m=0}^{\infty}(x/a)^{\nu+m}I_{\nu+m}(ax)$$
$$= \int_0^x \exp\left[-\frac{1}{2}(t^2+a^2)\right] t^\nu I_{\nu-1}(at)\,dt.$$

The particular case $\nu=1$ is given in Reference 2.

(3) As a final example we consider the evaluation of the integral

$$P = \int_0^\infty x^{-\nu+1}\exp\left[-\frac{1}{2}(x^2+b^2)\right] I_\nu(bx)$$
$$\cdot \int_0^{xt} y^{\nu+1}\exp\left[-\frac{1}{2}(y^2+a^2)\right] I_\nu(ay)\,dy\,dx,$$

the particular case in which $\nu=0$ having arisen in connection with a problem in noise theory. Differentiation of P with respect to t gives

$$P' = t^{\nu+1}\exp\left[-\frac{1}{2}(a^2+b^2)\right]\int_0^\infty x^3\exp\left[-\frac{1}{2}x^2(1+t^2)\right] I_\nu(bx)I_\nu(atx)\,dx$$

in which the integral may be obtained from

$$\int_0^\infty x\exp(-sx^2)J_\nu(ax)J_\nu(bx)\,dx$$
$$= (2s)^{-1}\exp\left[-\frac{1}{4}(a^2+b^2)/s\right] I_\nu\left(\frac{1}{2}ab/s\right)$$

by differentiation with respect to s.[3] The result, using (6), is

$$P' = 2(1+t^2)^{-2}t^{\nu+1}\exp\left[-\left(\frac{b^2t^2+a^2}{2(1+t^2)}\right)\right]$$
$$\cdot\left[\left(1-\nu+\frac{a^2t^2+b^2}{2(1+t^2)}\right)I_\nu\left(\frac{abt}{1+t^2}\right)+\frac{abt}{1+t^2}I_{\nu-1}\left(\frac{abt}{1+t^2}\right)\right].$$

Now if in (7) we substitute $f=bt/a$, $g=abt/(1+t^2)$ (it may be noted that f and g thus satisfy (8) with $\gamma/\alpha=b^2$ and $\gamma/\beta=a^2$) then differentiation gives

[3] Reference [4, p. 395, Equation (1)].

$$\left\{T_\nu \exp\left[-\frac{1}{2}(b^2t^2+a^2)/(1+t^2)\right]\right\}'$$
$$= \exp\left[-\frac{1}{2}(b^2t^2+a^2)/(1+t^2)\right](bt/a)^\nu(1+t^2)^{-2}$$
$$\times\left[abI_{\nu-1}(abt/(1+t^2))+a^2tI_\nu(abt/(1+t^2))\right]$$

from which it may shown, using (6), and comparing with P' as given directly above, that

$$\left\{\exp\left[-\frac{1}{2}(b^2t^2+a^2)/(1+t^2)\right]\left[T_\nu-(1+t^2)^{-1}(bt/a)^\nu I_\nu(abt/(1+t^2))\right]\right\}'$$
$$=\left(\frac{b}{a}\right)^\nu P'.$$

Thus, if Re $(\nu)>-1$,

$$P_\cdot=\left(\frac{a}{b}\right)^\nu \exp\left[-\frac{1}{2}(b^2t^2+a^2)/(1+t^2)\right]\sum_{m=0}^{\infty}\epsilon_m\left(\frac{bt}{a}\right)^{\nu+m}I_{\nu+m}\left(\frac{abt}{1+t^2}\right)$$

where

$$\epsilon_m=\begin{cases}t^2/(1+t^2) & \text{if } m=0,\\ 1 & \text{if } m>0.\end{cases}$$

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