DETERMINANTS OF HARMONIC MATRICES

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This paper is concerned with extensions of a theorem by H. S. Wall (Theorem 3 of [3]): if \( M \) is a 2\( \times \)2 harmonic matrix and \( F \) corresponds to \( M \) then \( \det M = 1 \) only in case \( F_{11} = -F_{22} \).

As in [3] let \( H_n \) denote the class of \( n \times n \) harmonic matrices and \( \Phi_n \) the class of \( n \times n \) matrices \( F \) of complex-valued functions from the real numbers, continuous and of bounded variation on every interval, such that \( F(0) = 0 \). In [3] Wall has shown that the Stieltjes integral equation,

\[
M(s, t) = I + \int_s^t dF(u) \cdot M(u, t),
\]

defines a one-to-one correspondence \( M \sim F \) between \( H_n \) and \( \Phi_n \). In studying this correspondence in a more abstract setting, the present author [1] has obtained the continuous product (or "product integral") representation,

\[
M(s, t) = \prod_s^t \{ I + dF \} \quad \text{when} \quad M \sim F.
\]

By using this representation, we now have the following extension of Wall's theorem cited above:

**Theorem 1.** If \( M \) is in \( H_n \) and \( M \sim F \) then

\[
\det M(s, t) = \text{Exp} \left( \sum_{i=1}^n [F_{pp}(t) - F_{pp}(s)] \right).
\]

**Proof.** Let \( f = \sum_i^n F_{pp} \), and \( g \) be the function from the ordered real number pairs \( \{s, t\} \) defined by:

\[
\det \{ I + F(t) - F(s) \} = 1 + f(t) - f(s) + g(s, t).
\]

Let \( J \) be a number interval, \( s \) and \( t \) numbers in \( J \), and \( b \) a positive number.

If \( u \) and \( v \) are real numbers then \( g(u, v) \) is a sum of products of \( n \) factors, of which at least two have the form \( F_{ij}(v) - F_{ij}(u) \). Thus there exist a natural number \( N \) and a sequence \( \{ \tau_p, h_p, k_p \} \) such that if \( u \) and \( v \) are numbers in \( J \) then

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1044
where each \( r_p \) is a bounded function from \( J \times J \) to the numbers, each \( h_p \) is one of the \( F_{ij} \), and each \( k_p \) is one of the \( F_{ij} \). From the uniform continuity of the \( h_p \) on \( J \) and the bounded variation of the \( k_p \) on \( J \), it follows that there exists a positive number \( c \) such that if \( \{ u_i \}^n_{i=0} \) is a monotone number sequence and \( u_0 = s \) and \( u_m = t \) and \( |u_i - u_{i-1}| < c \) for \( i = 1, \ldots, m \) then \( \sum_{i=1}^{m} |g(u_{i-1}, u_i)| < b \); hence,

\[
\prod_{i=1}^{m} \left\{ 1 + f(u_i) - f(u_{i-1}) \right\} - \det \prod_{i=1}^{m} \left\{ I + F(u_i) - F(u_{i-1}) \right\} \\
\leq \sum_{i=1}^{m} |g(u_{i-1}, u_i)| \exp \left( \sum_{i=1}^{m} |f(u_i) - f(u_{i-1})| \right) \\
+ \sum_{i=1}^{m} |f(u_i) - f(u_{i-1}) + g(u_{i-1}, u_i)| \\
\leq b \exp \left( b + 2 \sum_{i=1}^{m} \int_{s}^{t} |dF_{pp}| \right).
\]

Formula (3) is now apparent, since \( \prod_{t} \{ 1 + df \} = \exp (f[t] - f[s]) \).

**Remark.** The formula (3) is a generalization of the well-known exponential form of the Wronskian of a fundamental set of solutions for an \( n \)th order linear differential equation.

It seems natural to ask for a similar result in the case of quasi-harmonic matrices [2]—the statement that the \( n \times n \) matrix \( M \) is quasi-harmonic means that \( M \) is an \( n \times n \) matrix of complex-valued functions from the ordered pairs \{ \( s, t \) \} of real numbers, which, for each \( t \), are of bounded variation in \( s \) on every interval and which are quasi-continuous in \( t \) for each \( s \), and that, for each ordered triple \{ \( r, s, t \) \} of real numbers, \( M(r, s) - M(s, t) = M(r, t) \) and

\[
M(s, s) = \frac{1}{2} [M(s-, s) + M(s, s-)] \\
= \frac{1}{2} [M(s, s+) + M(s+, s)] = I.
\]

Let \( QH_n \) denote the class of \( n \times n \) quasi-harmonic matrices and \( Q\Phi_n \) the class of \( n \times n \) matrices \( F \) of complex-valued functions from the real numbers, of bounded variation on every interval, such that

\[
[F(r) - F(r-)]^2 = [F(r+) - F(r)]^2 = F(0) = 0 \quad \text{for each} \ r.
\]
In [2] we have shown that (1), with mean integrals replacing the Stieltjes integrals used by Wall, defines a one-to-one correspondence \( M \sim F \) between \( QH_n \) and \( \Phi_n \) which extends the correspondence established by Wall in [3] and which is also determined by (2).

**Theorem 2.** If \( M \) is in \( QH_n \) and \( M \sim F \) and \( G \) is the "continuous part" of \( F \) then

\[
\det M(s, t) = \exp \left( \sum_{1}^{n} \left[ G_{pp}(t) - G_{pp}(s) \right] \right).
\]

**Proof.** By the "continuous part" of \( F \) we mean (as in proof of Theorem 2.4 of [2]) an element \( G \) of \( \Phi_n \) such that, if \( r_1, r_2, \cdots \) is a simple number sequence such that if \( s \) is a number at which \( F \) is not continuous then there is a natural number \( k \) such that \( r_k = s \), there is a sequence \( F_1, F_2, \cdots \) of elements of \( QH_n \) such that

(i) \( F_1 = G \) and, if \( j \) is a natural number and \( [a, b] \) is a number interval which does not contain \( r_j \), then \( F_{j+1}(b) - F_{j+1}(a) = F_j(b) - F_j(a) \) and \( F_{j+1}(r_j) - F_{j+1}(r_j-) = F(r_j) - F(r_j-) \) and \( F_{j+1}(r_j+) - F_{j+1}(r_j) = F(r_j+) - F(r_j) \), and

(ii) \( F_k(s) \to F(s) \) as \( k \to \infty \) for each real number \( s \).

Let \( F_1, F_2, \cdots \) be such a sequence and \( M_k(s, t) = \prod I^i \{ I + dF_k \} \) for each natural number \( k \).

If \( A \) is an \( n \times n \) matrix of complex numbers and \( A^2 = 0 \) then \( \det \{ I + A \} = 1 \). This may easily be seen as follows: let \( P \) be the function from the complex numbers defined by \( P(z) = \det \{ I + zA \} \); now \( P \) is a polynomial satisfying the identity \( P(z)P(-z) = 1 \), so that \( P \) has no zero; hence \( P \) is constant and its only value is \( P(0) \), which is 1. Thus we see that \( \det M_k(s, t) = \det M_1(s, t) \) for \( k = 1, 2, \cdots \). The formula (6) now follows from Theorem 2.5 of [2] and Theorem 1 of the present paper.

**Bibliography**