SHEAVES AND DIFFERENTIAL EQUATIONS

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Introduction. The theory of sheaves was originated by Leray and subsequently used by H. Cartan, Kodaira, Serre, and others in order to pass from a local to a global situation. For this reason it seems natural to apply this theory to differential equations, where much is known about the local theory and little about global problems. The present paper is an attempt to define and give some basic properties of the sheaves that are related to differential equations.

Let M be a smooth domain in some euclidean space and let D be a linear partial differential operator whose coefficients are C^∞ on M. We define the sheaf A on M by requiring that, for each $x \in M$, the stalk A_x of A at x consists of all functions f which are defined and C^∞ in a neighborhood of x and satisfy $Df=0$ in this neighborhood of x. We consider the cohomology groups $H^j(A)$ of M with coefficients in A. We show that, if D has constant coefficients, or if D is elliptic or hyperbolic, then $H^j(A)=0$ for $j \geqq 2$. Moreover, if $\mathcal{E}$ denotes the space of C^∞ functions on M, then $H^1(A) \approx \mathcal{E}/D\mathcal{E}$, so $H^1(A)$ measures how many functions in $\mathcal{E}$ are not of the form Df for $f \in \mathcal{E}$.

Instead of using C^∞ functions, we can use distributions and obtain a sheaf B instead of A. Of course, if D is elliptic, $B=A$.

We can use the theory of sheaves to study a very general type of boundary value problem for differential equations. The vanishing of a certain cohomology group gives a necessary and sufficient condition in order that all "admissible" boundary values have global solutions.

We can also consider differential equations on a C^∞, or analytic, manifold. The most natural operator to use in that case is the Laplacian. The dimensions of the cohomology groups we obtain are easily interpreted in terms of the Betti numbers of the manifold.

For the definitions and general properties of sheaves and the cohomology of a space with coefficients in a sheaf, we refer to Cartan's seminar notes (see [**1**]) or to Serre's paper [**10**].

1. The sheaves associated with a differential operator. Let M be a subspace of an indefinitely differentiable manifold R. Let f be a function which is defined in a neighborhood in M of a point $x \in M$. We say that f is C^∞ at x if f can be extended to a C^∞ function in a neighborhood of x in R. We say that D is a *differential operator* on M if

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a. For each $x \in M$, D defines a linear map D_x of the space of C^∞ functions at x into itself.

b. If f is in C^∞ at all points of a set U in M, then for each $x, x' \in U$, $D_x f = D_{x'} f$.

c. If f, g are in C^∞ at $x \in M$ and if $g(y) = 1$ for y in a neighborhood of x in M, then $(D_x fg)(x) = (D_x f)(x)$.

It is clear that if M is an open subset of R, then every partial differential operator with C^∞ coefficients on M defines a differential operator on M. But we allow the possibility of defining a differential operator in different ways in various parts of M.

For each $x \in M$ let E_x be the vector space of all C^∞ functions at x, two such functions being considered equal if they coincide on a neighborhood of x. The E_x form a sheaf E on M. For any $f \in E_x$, there exists a neighborhood N of x in M such that f defines elements $f_y \in E_y$ for all $y \in N$. These sets $\{f_y\}_{y \in N}$ form a basis for the open sets in E. It is clear that E is locally homeomorphic to M. For the other sheaves which we shall consider below, the topology is defined similarly. We define the sheaf A on M by requiring that, for each $x \in M$, A_x consists of all $f \in E_x$ which satisfy $D_x f = 0$ in E_x. By condition (c), this condition is compatible with the equivalence relation in E_x. The sheaf DE is defined by $(DE)_x = \{D_x f\}_{f \in E_x}$ for any $x \in M$. We have the exact sequence of sheaves:

$$0 \to A \to E \xrightarrow{D} DE \to 0. \tag{1}$$

From (1) we deduce the exact sequence of cohomology groups (we write $H^i(A)$ for $H^i(M, A)$, etc.)

$$\begin{aligned} 0 \to H^0(A) \to H^0(E) \xrightarrow{D} H^0(DE) \to H^1(A) \to H^1(E) \to H^1(DE) \\ \to H^2(A) \to H^2(E) \to \cdots . \end{aligned} \tag{2}$$

Now, E is a fine sheaf (faisceau fin) so that $H^i(E) = 0$ whenever $i \geqq 1$ (see [1]). We have thus

THEOREM 1. *For any* $j \geqq 1$, $H^j(DE) \approx H^{j+1}(A)$. *Moreover*, $H^1(A) \approx H^0(DE)/DH^0(E)$.

Now, it is clear that $H^0(E) = \mathcal{E}$ is just the space of C^∞ functions on M. Moreover, $H^0(A)$ is the subspace of $\mathcal{E}$ consisting of all C^∞ functions f on M for which $Df = 0$ (i.e. $D_x f = 0$ for all $x \in M$). $H^0(DE)$ consits of all $f \in \mathcal{E}$ which are locally of the form Dg, i.e. for each $x \in M$, there is a function g which is C^∞ at x and satisfies $D_x g = f$ in a neighborhood of x. We have thus

COROLLARY. *A necessary and sufficient condition for* $H^1(A)=0$ *is that every function which is locally of the form* Dg *is also globally of the form* Dg.

DEFINITION. D is called *fine* if $DE=E$.

Examples of fine operators are (if M is a "smooth" subspace of R):

1. Elliptic partial differential operators (see [**8**]).
2. Hyperbolic partial differential operators (see [**6**]).
3. Partial differential operators with analytic coefficients (by Cauchy-Koweleski), if R is a real analytic manifold.

THEOREM 2. *If* D *is fine, then* $H^1(A)=0$ *if and only if* $D\mathcal{E}=\mathcal{E}$; *for any* $j\geqq 2$, $H^j(A)=0$.

If M is a smooth domain, then $D\mathcal{E}=\mathcal{E}$ if D is elliptic (see [**8**]) or if D has constant coefficients (see [**3**; **8**]).

We can use the above to formulate a Cousin-type problem for differential operators: Suppose that we are given a locally finite open covering $\{U_i\}$ of M and, for each i a function f_i which is defined and C^∞ on U_i. Suppose that, for any i, j, $D(f_i-f_j)=0$ on $U_i\cap U_j$. Does there exist an $f\in\mathcal{E}$ such that, for every i, $D(f-f_i)=0$ on U_i? We call this type of problem a *D-Cousin problem.*

Now, set $h_{ij}=f_i-f_j$ in $U_i\cap U_j$. Then it is easily seen that $\{h_{ij}\}$ defines a 1-cocycle of the sheaf A for the open covering $\{U_i\}$ and, by taking refinements of $\{U_i\}$, and passage to the direct limit, $\{h_{ij}\}$ defines an element $h\in H^1(A)$. The condition that the given D-Cousin problem be solvable is that $h=0$. Thus we have

THEOREM 3. *If* $H^1(A)=0$, *then every D-Cousin problem is solvable.*

Thus, if M is a "smooth" domain, then the D-Cousin problem is solvable if D is elliptic or has constant coefficients.

For any $x\in M$ we shall say that S is a distribution at x if we can find a function g which is C^∞ on R such that $g=1$ on a neighborhood of x in R and such that the map $f\to S\cdot gf$ for f a C^∞ function on R of compact carrier (support) defines a distribution on R.

DEFINITION. The differential operator D on M is called *strong* if for every $x\in M$, we can find a function g which is C^∞ on R, such that $g=1$ on a neighborhood of x in R, and such that, for any distribution S at x, the map $f\to S\cdot D_xgf$ for f a C^∞ function on R of compact carrier, defines a distribution on R. (We write D_xS for this distribution at x.)

Just as in the case of C^∞ functions, the distributions form a sheaf D' on M whose stalk at any point x is denoted by D'_x. Assume D is strong and, for each $x\in M$, set B_x as the set of all distributions at x for which $D_xB_x=0$ in a neighborhood of x. The $\{B_x\}$ form, in an

obvious manner, the stalks of a sheaf B. The sheaf DD' is defined by $(DD')_x = \{DS\}_{S \in D'_x}$. Then we have the exact sequence

$$0 \to B \to D' \xrightarrow{D} DD' \to 0 \tag{3}$$

from which we deduce as above

THEOREM 4. *For any* $j \geqq 1$, $H^j(DD') \approx H^{j+1}(B)$. *Moreover*, $H^1(B) \approx H^0(DD')/DH^0(D')$.

DEFINITION. D is called *ultrafine* if $DD' = D'$.

We obtain as above

THEOREM 5. *If D is ultrafine, then $H^1(B) = 0$ if and only if $D\mathfrak{D}' = \mathfrak{D}'$, where $\mathfrak{D}' = H^0(D')$ is the space of distributions on M; also $H^j(B) = 0$ for $j \geqq 2$.*

It has been shown (see [**4**]) that if D has constant coefficients and if M is "smooth," then $D\mathfrak{D}' = \mathfrak{D}'$ so that $H^1(B) = 0$.

We consider E as a subsheaf of D' in the usual manner, so A is a subsheaf of B.

DEFINITION. D is called *elliptic* if $A = B$.

If M is "smooth," then conditions for ellipticity are given in [**5**] and in [**8**].

THEOREM 6. *If D is elliptic, fine, and ultrafine, then $\mathcal{E}/D\mathcal{E} \approx \mathfrak{D}'/D\mathfrak{D}'$.*

In general (see [**3**; **8**]) it is easier to prove that $\mathcal{E} = D\mathcal{E}$ than that $\mathfrak{D}' = D\mathfrak{D}'$; by the above, the conclusion $\mathfrak{D}' = D\mathfrak{D}'$ follows from $E = DE$ for elliptic operators.

Now, assume that R is a complex analytic manifold and denote by $\mathcal{H}$ the sheaf of germs of holomorphic functions on M. Since $\mathcal{H}$ is a subsheaf of E, D is defined on $\mathcal{H}$.

DEFINITION. D is called *regular* if $D\mathcal{H}_x \subset \mathcal{H}_x$ for all $x \in M$.

Assume now that D is regular. The subsheaf C of $\mathcal{H}$ is defined by: For $f \in \mathcal{H}_x$, $f \in C_x$ if $D_x f = 0$. The subsheaf $D\mathcal{H}$ of $\mathcal{H}$ is defined by $(D\mathcal{H})_x = \{Df\}_{f \in \mathcal{H}_x}$ for any $x \in M$. We have the exact sequence of sheaves

$$0 \to C \to \mathcal{H} \xrightarrow{D} D\mathcal{H} \to 0 \tag{5}$$

from which we deduce the cohomology sequence

$$\begin{aligned} &0 \to H^0(C) \to H^0(\mathcal{H}) \xrightarrow{D} H^0(D\mathcal{H}) \to H^1(C) \to H^1(\mathcal{H}) \to H^1(D\mathcal{H}) \\ &\quad \to H^2(C) \to \cdots . \end{aligned} \tag{6}$$

Suppose that M is a Stein manifold (see [2]) so that $H^j(\mathcal{H})=0$ for $j\geqq 1$. Then we deduce from (6)

THEOREM 7. *If M is a Stein manifold, then for any $j\geqq 1$, $H^j(D\mathcal{H})\approx H^{j+1}(C)$. Moreover, $H^1(C)\approx H^0(D\mathcal{H})/DH^0(\mathcal{H})$.*

COROLLARY. *If M is a Stein manifold, then a necessary and sufficient condition for $H^1(C)=0$ is that every holomorphic function on M which is locally of the form Dg be globally of the form Dg.*

Just as in the case of C^∞ functions, we can formulate an *analytic D-Cousin* problem for M. Then we deduce

THEOREM 8. *For any M, if $H^1(C)=0$, then the analytic D-Cousin problem is solvable for M.*

2. **Boundary value problems.** We assume that R is an indefinitely differentiable manifold and let $\dot{M}$ be a distinguished subset of M; let K be a sheaf on $\dot{M}$ which is a subsheaf of the sheaf defined by the C^∞ functions on $\dot{M}$. The sheaf G on R is defined as follows: For $x\in M-\dot{M}$, $G_x=A_x$; for $x\in\dot{M}$, $f\in G_x$ if $f\in A_x$ and if the restriction of f to a suitable neighborhood of x in $\dot{M}$ lies in K. Denote by L the quotient sheaf: $L=A/G$. It is clear that $L_x=0$ for any $x\in\dot{M}-M$, so, as far as cohomology theory is concerned, we may consider L as a sheaf over $\dot{M}$; thus, we shall write $H^j(L)$ to denote the j cohomology group of either $\dot{M}$ or $\dot{M}$ with coefficients in L.

We have the exact sequence of sheaves

$$0\to G\to A\xrightarrow{j}L\to 0 \tag{7}$$

from which we deduce the exact sequence of cohomology groups

$$\begin{aligned}0\to H^0(G)\to H^0(A)\xrightarrow{j}H^0(L)\to H^1(G)\to H^1(A)\to H^1(L)\\ \to H^2(G)\to\cdots.\end{aligned} \tag{8}$$

We deduce from (8) and Theorem 2:

THEOREM 9. *If $H^1(G)=0$, then $H^0(L)\approx jH^0(A)$. If D is fine, then for $i\geqq 1$, $H^i(L)\approx H^{i+1}(G)$.*

Let us interpret Theorem 8 in classical language: $\dot{M}$ is the "boundary" of M; the sheaf K is the "boundary conditions" for D. G consists of all (local) solutions of the equation $Df=0$ which satisfy the given boundary conditions. L consists of those local C^∞ functions on $\dot{M}$ which can be extended (locally) to solutions of $Df=0$; L may be con-

sidered the sheaf of "admissible boundary values." Thus the first statement of Theorem 9 may be interpreted as

COROLLARY 1. *If $H^1(G)=0$, then every admissible boundary value on $\dot{M}$ is the restriction to $\dot{M}$ of a global solution of the equation $Df=0$.*

The second statement of Theorem 9 gives:

COROLLARY 2. *Suppose every boundary value is admissible, that is, suppose that L is the sheaf of all C^∞ functions on $\dot{M}$. Then $H^i(G)=0$ for $i\geqq 2$. Also, if L consists of 0 only, then $H^i(G)=0$ for $i\geqq 2$.*

Now, let us look for another interpretation of the cohomology groups of G. We have the exact sequence

$$0\to G\to E\xrightarrow{k} E/G\to 0. \tag{9}$$

Since G is a subsheaf of A, $E/A=DA$ may be considered as a subsheaf of E/G. Thus, if D is fine, E/G is also fine, and we deduce from (9) and Theorem 9

THEOREM 10. *If D is fine, then $H^i(L)=H^{i+1}(G)=0$ for $i\geqq 1$. Moreover, $H^1(G)\approx H^0(E/G)/k\mathcal{E}$.*

Now, $(E/G)_x=(E/A)_x=(DE)_x$ if $x\notin\dot{M}$, while, for $x\in\dot{M}$, $(E/A)_x=(DE)_x$ may be considered as a subspace of $(E/G)_x$. Thus, if D is fine, we must have $\mathcal{E}=H^0(DE)=H^0(E/A)\subset H^0(E/G)$. Thus, if E/G could be identified with a subsheaf of E, it would follow that $\mathcal{E}=H^0(E/G)$. In that case also, $k\mathcal{E}$ would be a space containing $D\mathcal{E}$ so we have

COROLLARY. *Suppose that D is fine and that E/G can be identified with a subsheaf of E. Then $H^1(A)=0$ implies $H^1(G)=0$.*

EXAMPLE 1. Let R be a Euclidean space of real dimension n; let M be the unit solid sphere in R (that is, the set of $x\in R$ with $|x|\leqq 1$), and let $\dot{M}$ be the boundary of M. Let D be the Laplacian on R, $D=\sum\partial^2/\partial x_k^2$, and let K be the sheaf on $\dot{M}$ consisting of 0 only. Then A is the sheaf of germs of harmonic functions on M, and G is the sheaf of germs of harmonic functions on M which are zero on $\dot{M}$. L may be identified with the sheaf of germs of C^∞ functions on $\dot{M}$. Thus, by Theorem 9, $H^i(G)=0$ for $i\geqq 2$. By the known results on the Dirichlet problem for M, it follows that j maps $H^0(A)$ onto $H^0(L)$ (j is just the restriction map). Thus, $H^1(G)=0$ also. Hence, $H^1(A)=0$ by (8) so, by Theorem 2, we have $\mathcal{E}=D\mathcal{E}$ (a result which is contained in [**3**]).

EXAMPLE 2. R, M, $\dot{M}$, and K are defined as in Example 1, except

that we require that $n=2$; D is $\partial/\partial\bar{z}$ where $z=x_1+ix_2$, x_1 and x_2 being the coordinates on R. A is the sheaf of germs of holomorphic functions on $M-\dot{M}$ which are C^∞ on M. Since any holomorphic function on M which is zero on $\dot{M}$ must be zero identically, $H^0(G)=\{0\}$. L may be identified with the sheaf of germs of C^∞ functions on $\dot{M}$ which can be extended to holomorphic functions in an open subset of $M-\dot{M}$. Now it is clear that there exist C^∞ functions on L which are the boundary values of functions which are holomorphic in a neighborhood of L in $M-\dot{M}$, but which are not boundary values of functions holomorphic in all of $M-\dot{M}$. Thus, by Corollary 1 of Theorem 9, $H^1(G)\neq 0$. Now, by Theorem 2 and the results of [**3**], $H^1(A)=0$. Thus, by (8) and our above remarks, $H^1(G)$ is even infinite dimensional. By Theorem 9, $H^i(L)=H^{i+1}(G)=0$ for $i\geqq 1$. Note that L is *not* a fine sheaf, so that the result $H^i(L)=0$ for $i\geqq 1$ does not seem to be trivial.

EXAMPLE 3. M is defined as in Example 1 above, M^* is the boundary of M and $\dot{M}$ is a neighborhood of M^* in M. Let D be the rth power of the Laplacian on R, where r is a positive integer. The sheaf K on $\dot{M}$ is defined as follows: $K_x=E_x$ for $x\in\dot{M}-M^*$; for $x\in M^*$, K_x consists of all germs of C^∞ functions at x which are zero on M^* together with all partial derivatives of order $\leqq r-1$. A is the sheaf of germs of r-harmonic functions on M which vanish "to the order $r-1$" on M^*. L may be identified with the germs of $p(r)$-tuples of C^∞ functions on M^*, where $p(r)$ is the number of differential operators $\partial^q/\partial x_1^{q_1}\cdots\partial x_n^{q_n}$, for $0\leqq q=\sum q_k\leqq r-1$ (because D is fine). By known results on the Dirichlet problem for M, it follows that j maps $H^0(A)$ onto $H^0(L)$ (j is the map which assigns to any $f\in H^0(A)$ the $p(r)$-tuple $\partial^q f/\partial x_1^{q_1}\cdots\partial x_n^{q_n}$ on M^*. Thus, $H^1(G)=0$, hence also $H^1(A)=0$ by (8); thus, by Theorem 2 we obtain again $\mathcal{E}=D\mathcal{E}$ (see [**3**]).

It is obviously possible to extend the above examples, e.g. to study Neuman's problem or other boundary value problems from the above point of view.

3. **The Laplacian on a compact manifold.** Suppose that M is a compact C^∞ manifold, and let D be the Laplacian on M. It is known (see [**8**]) that D is fine. It follows easily from results of DeRham and Kodaira (see [**9**]) that $\mathcal{E}/D\mathcal{E}$ may be identified with the space of harmonic functions on M so that by Theorem 1 dim $H^1(A)=b_0$ is the zero Betti number of M. If we denote by E^r the sheaf of germs of C^∞ r-forms on M and define A^r by requiring that

$$0\to A^r\to E^r\to DE^r\to 0 \tag{10}$$

be exact, then reasoning as in the proof of Theorem 1 we deduce

THEOREM 11. *For any* r, $H^j(A^r)=0$ *for* $j\geqq 2$. *Moreover*, $\dim H^1(A^r)=\dim H^0(A^r)=b_r$ *is the rth Betti number of* M.

We can use a similar analysis on compact complex analytic manifold, where we can also apply our methods to forms with values in complex line bundles (see [7]) or more general types of bundles (see [11]).

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