ON THE LEFT NUCLEUS OF A BRÜCK RING

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Let $F$ be a field of characteristic two and let $R$ be the set of all couples $(f, g)$ with $f$ and $g$ in $F$. Define equality and addition in $R$ componentwise, and define multiplication by

\[(f, g)(h, k) = (fh + gk, fk + gh),\]

where $\theta$ is an additive endomorphism of $F$. $R$ is a right alternative ring which is alternative if and only if $k\theta = km$, for all $k$ in $F$ and for some fixed $m$ in $F$. $R$ is a division ring if and only if, for every $f$ in $F$, the mapping $\mu(f)$, defined by

\[(1.2) \quad x\mu(f) = x\theta + xf^2, \quad \text{all } x \text{ in } F,\]

is one-to-one of $F$ upon $F$. (See [2], Appendix.) R. H. Brück has shown that fields $F$ having additive endomorphisms $\theta$ satisfying (1.2), and yet not right multiplications, actually exist. The author, in [1], has therefore called a not alternative, right alternative division ring $B$ (necessarily of characteristic two) a Brück ring if it is two dimensional over some field $F$ and if multiplication in $B$ is given by (1.1). The field $F$ turns out to be the left nucleus of $B$ and, in all the examples given by Brück, is a simple transcendental extension of a field of characteristic two, and thus imperfect.

In this note we give an elementary proof that this situation must of necessity occur:

**Theorem.** The left nucleus of a Brück ring is always an imperfect field.

**Proof.** If $B$ is a Brück ring with left nucleus $F$, assume that $F$ is perfect. Let $\theta$ be the additive endomorphism of (1.1) and suppose that $x\theta = y$, $x \neq 0$ in $F$. Then $y \neq 0$ and $y = xz$, some $z$ in $F$. But then $z = v^2$, some $v$ in $F$, so that $x\theta = xv^2$ and $x\theta + xv^2 = 0$. However, $0\theta + 0v^2 = 0$ and thus the mapping $\mu(v)$, defined by (1.2), is not one-to-one of $F$ upon $F$. Hence $B$ is not a division ring, a contradiction. It must therefore be true that $F$ is imperfect, which is what we wanted to prove.

We remark that the following corollary answers a question raised by Kleinfeld in a letter to the author dated May, 1953 and mentioned by the author in [1] as an unsolved problem.

**Corollary.** There exist no finite Brück rings.

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1 Numbers in brackets refer to the bibliography.
THE EXISTENCE OF OUTER AUTOMORPHISMS OF SOME GROUPS

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F. Haimo and E. Schenkm an [1] raised the question: Does a nilpotent group $G$ always possess an outer automorphism? The answer is in the affirmative if $G$ is finite and nilpotent of class 2, as is seen from a Schenkm an's [1] stronger result. The object of this note is to show that the answer is also in the affirmative for another family of nilpotent groups, namely the family of all finite $p$-groups $G$ of order greater than 2 such that $x^p = e$ for every element $x$ in $G$. Actually, our result is somewhat stronger:

**Theorem 1.** Suppose that $G$ is a group every element of which is of order a divisor of a fixed integer $n > 1$. If $G$ has a normal subgroup $N$ such that the factor group $G/N$ is cyclic of order $n$ and such that the intersection $N \cap Z$ of $N$ with the center $Z$ of $G$ contains an element $a_0$ of order $n$, then $G$ possesses an outer automorphism which induces identity automorphisms on both $N$ and $G/N$.

For the proof we need the following

**Lemma.** If elements $a, a_0, a_1, \ldots, a_k$ in a group $G$ are such that $aa_0 = a_0a, a_ia_j = a_0a$ for $i, j = 0, 1, \ldots, k$ and such that

\[
\begin{align*}
(1) & \quad a_ia_i^{-1} = a_{i-1} \quad (i = 1, 2, \ldots, k) \\
(2) & \quad (aa_k)^r = a a_{k-1} \cdots a_{k-r+1}
\end{align*}
\]

then we have, for $r = 1, 2, \ldots$,

\[
(2) \quad (aa_k)^r = a a_{k-1} \cdots a_{k-r+1}
\]

where we set $a_{-1} = a_{-2} = \cdots = e$. If, moreover, $a_0$ is of order $n$ and if $a_i^n = e$ for $i = 1, 2, \ldots, k$, then

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