ON A TEST FOR THE CONVERGENCE OF FOURIER SERIES

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Suppose that \( \phi(t) \) is an even and integrable function, periodic with period \( 2\pi \). Let

\[
\phi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt.
\]

Wang\(^1\) has established the following two theorems.

**Theorem A.** If

(i) \[
\int_0^t \phi(u) du = o \left( \frac{t}{\log (1/t)} \right) \quad (t \to 0),
\]

(ii) \[
a_n > - K \frac{\log n}{n}
\]

for some \( K > 0 \), then the series \( \sum a_n \) converges.

**Theorem B.** There exists an even function \( \phi(t) \), satisfying (i), whose Fourier series diverges at \( t=0 \), while

(ii\(^\prime\)) \[
a_n = o \left( \frac{(\log n)^2}{n} \right).
\]

The problem of bridging the gap between Theorems A and B is still unsolved. In this note, we remove this gap by proving the following

**Theorem.** There exists an even function \( \phi(t) \), satisfying (i), whose Fourier series diverges at \( t=0 \), while

(ii\(^\prime\)) \[
a_n = o \left( \frac{(\log n)^{1+\eta}}{n} \right),
\]

where \( \eta \) is any preassigned positive value.

**Proof.** We choose three sequences of positive values \( (\lambda_i) \), \( (\alpha_i) \) and \( (\epsilon_i) \) such that

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0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \rightarrow \infty;\text{ }^2
\alpha_i = \log \frac{\lambda_i}{\lambda_{i-1}}, \quad (i \geq 1), \alpha_0 = \pi;
\gamma_i = (\log \frac{\alpha_{i-1}}{\alpha_i})^{-1/(1+\epsilon)}.

And define an even function:
\phi(t) = c_i \sin \lambda_i t \quad (\alpha_i < t \leq \alpha_{i-1}; \ i = 1, 2, 3, \cdots).

Then, \phi(t), being bounded, is integrable in the Lebesgue sense.

By a number of direct calculations,\textsuperscript{3} we can arrive at the following results:

\begin{align*}
(1) \quad \frac{1}{t} \log \frac{1}{t} \int_0^t \phi(u) du &= \frac{1}{\alpha_n} \log \frac{1}{\alpha_n} O \left( \sum_{i<n} \frac{c_i}{\lambda_i} \right) \text{ for } \alpha_n < t \leq \alpha_{n-1}.
\end{align*}

Let the \lambda_nth partial sum of the Fourier series of \phi(t) at \ t=0 be \ S_{\lambda_n}, then

\begin{align*}
S_{\lambda_n} &= \frac{2}{\pi} \int_0^\pi \phi(t) \frac{\sin \lambda_n t}{t} dt + o(1)
\end{align*}

\begin{align*}
&= \frac{2}{\pi} \left( \sum_{i<n} + \sum_{i\geq n} \right) c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt \\
&\quad + \frac{2c_n}{\pi} \int_{\alpha_n}^{\alpha_{n-1}} \frac{\sin^2 \lambda_n t}{t} dt + o(1)
\end{align*}

\begin{align*}
&= \frac{2}{\pi} \left( \Sigma_1 + \Sigma_2 + \Sigma_3 \right) + o(1).
\end{align*}

We have

\begin{align*}
(2) \quad \Sigma_1 &= \sum_{i<n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt \\
&= O \left( \sum_{i<n} \frac{c_i}{\alpha_i (\lambda_n - \lambda_i)} \right) = \frac{1}{\lambda_n} O \left( \sum_{i<n} \frac{c_i}{\alpha_i} \right),
\end{align*}

if \lambda_{n-1}/\lambda_n < k < 1 for every \ n.

\begin{align*}
(3) \quad \Sigma_2 &= \sum_{i\geq n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt = \lambda_n O \left( \sum_{i\geq n} \frac{c_i \alpha_{i-1}}{\lambda_i \alpha_i} \right),
\end{align*}

\textsuperscript{3} We take the sequence (\alpha_i) in such a manner that \alpha_i < \alpha_{i-1} for every \ i and \ c_i = o(1) as \ i \rightarrow \infty.

since the total variation of \( \sin \lambda_n^t \) in \( (\alpha_i, \alpha_{i-1}) \) is \( O(\alpha_{i-1} \lambda_n^t) \).

(4) \[ S_2 = c_n \int_{\alpha_n}^{\alpha_n-1} \sin^2 \lambda_n^t \, dt = \frac{c_n}{2} \log \frac{\alpha_{n-1}}{\alpha_n} + O(\frac{c_n}{\alpha_n \lambda_n^t}). \]

Finally, we have

\[
a_n = \frac{2}{\pi} \int_0^\pi \phi(t) \cos ntdt = \frac{2}{\pi} \left( \sum_{i \leq \rho_n} + \sum_{i > \rho_n} \right) c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i \cos nt \, dt
\]

say, where \((\rho_n)\) is a rapidly increasing sequence of positive integers. Now,

(5) \[ \sum_{i \leq \rho_n} J_i = \sum_{i \leq \rho_n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i \cos nt \, dt = \frac{1}{n} O \left( \sum_{i \leq \rho_n} \lambda_i \alpha_{i-1} \right). \]

On the other hand,

(6) \[ \sum_{i > \rho_n} J_i = O \left( \sum_{i > \rho_n} \frac{c_i}{|\lambda_i - n|} \right). \]

In order to establish the theorem, we need to make the value in (1), \( o(1), \Sigma_1 + \Sigma_2 = o(1), \Sigma_3 > 1 \) and the values of the expressions in (5) and (6) are both \( o((\log n)^{1+\gamma}/n) \).

We take \( \lambda_i = \left(\frac{(i+1)!}{i!}\right) \). Then

\[
\frac{1}{t} \log \frac{1}{t} \int_0^t \phi(u) \, du = \left(\frac{(i+1)!}{i!}\right) o \left( \sum_{i \in \mathbb{Z}} \frac{1}{((i+1)!)} \right) = o(1)
\]

as \( t \to +\infty \). And

\[
\Sigma_1 = \frac{1}{((n+1)!)^x} \log \frac{(i+1)!}{(i+1)!} = \frac{1}{((n+1)!)^x} \log \frac{(n!)^y}{\log n} = o(1).
\]

\[
\Sigma_2 = \left(\frac{(n+1)!}{(i!)^x}\right) o \left( \sum_{i > \rho_n} \frac{1}{(i!)^x} \right) = o(1).
\]

\[
\Sigma_3 \approx \frac{n}{2(\log (n+1))^{1/(1+\gamma)}} \log \frac{(n+1)!}{n!} + o(1)
\]

\[
= \frac{n}{2} \left( \log (n+1) \right)^{1/(1+\gamma)} + o(1) > 1.
\]
This shows that the Fourier series of \( \phi(t) \) diverges at \( t=0 \).

Moreover,

\[
\sum_{i \leq p_n} J_i = \frac{1}{n} o \left( \sum_{i \leq p_n} \frac{(i+1)^{1/2} \log i!}{(i!)^{1/2}} \right) = \frac{1}{n} o(p_n^{1/2} \log p_n),
\]

\[
\sum_{i > p_n} J_i = o \left( \sum_{i > p_n} \frac{1}{((i+1)!)^{1/2} - n} \right) = o \left( \sum_{i > p_n} \frac{1}{((i+1)!)^{1/2} - (i!)^{1/2}} \right),
\]

since \((i!)^{1/2} \geq \log p_n > n\). Therefore,

\[
\sum_{i > p_n} J_i = o \left( \frac{1}{(p_n!)^{1/2}} \right) = o \left( \frac{e^{p_n}}{p_n^{(n+1)/2}} \right)
\]

by Stirling's formula

\[
e^{n!} \cong (2\pi)^{1/2} n^{n+1/2}.
\]

Now, if we choose a rapidly increasing sequence of positive integers \((n_i)\) such as \( n_i = ((i+1)! \) and define another function \( \phi^*(t) \) from \( \phi(t) \) as follows:

\[
\phi^*(t) = c^*_t \sin \lambda^*_t t \quad (\alpha^*_t < t \leq \alpha^*_{i-1}),
\]

where \( \lambda^*_t = \lambda_t \), \( \alpha^*_t = \alpha_t \) and \( c^*_t = c_n \) if \( \nu = n_i \) for a certain \( i \); and \( c^*_t = 0 \) if \( \nu \neq n_i \) for every \( i \), under this circumstance, the corresponding condition (i) of the theorem is still satisfied and the Fourier series of \( \phi^*(t) \) still diverges at \( t=0 \) as before. But, by this method of choice of the sequence \((n_i)\), we can easily find that the corresponding

\[
\sum_{i \leq p_n} J^*_i = \frac{1}{n} o(p_n^{1/2} \log p_n).
\]

Taking \( p_n = [\log n (\log^2 n)^{-1/10}] \), we obtain

\[
\sum_{i \leq p_n} J^*_i = o \left( \frac{(\log n)^{1/2}}{n} \right),
\]

also,

\[
\sum_{i > p_n} J^*_i = o \left( \frac{(\log n)^{1/2}}{n} \right).
\]

Thus, \( a^*_t = o((\log n)^{1/2}/n) \). This completes the proof of the theorem.

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