ON A TEST FOR THE CONVERGENCE OF FOURIER SERIES

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Suppose that $\phi(t)$ is an even and integrable function, periodic with period 2π. Let

$$\phi(t) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} a_\nu \cos \nu t.$$

Wang[1] has established the following two theorems.

THEOREM A. *If*

$$\text{(i)} \qquad \int_0^t \phi(u)du = o\left(\frac{t}{\log (1/t)}\right) \qquad (t \to +0),$$

$$\text{(ii)} \qquad a_n > -K \frac{\log n}{n}$$

for some $K>0$, *then the series* $\sum a_n$ *converges.*

THEOREM B. *There exists an even function* $\phi(t)$, *satisfying* (i), *whose Fourier series diverges at* $t=0$, *while*

$$\text{(ii}'\text{)} \qquad a_n = o\left(\frac{(\log n)^2}{n}\right).$$

The problem of bridging the gap between Theorems A and B is still unsolved. In this note, we remove this gap by proving the following

THEOREM. *There exists an even function* $\phi(t)$, *satisfying* (i), *whose Fourier series diverges at* $t=0$, *while*

$$\text{(ii}''\text{)} \qquad a_n = o\left(\frac{(\log n)^{1+\eta}}{n}\right),$$

where η *is any preassigned positive value.*

PROOF. We choose three sequences of positive values (λ_i), (α_i) and (c_i) such that

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[1] Proc. London Math. Soc. (2) vol. 47 (1942) p. 308.

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty;^2$$

$$\alpha_i = \log \lambda_i/\lambda_i \qquad (i \geqq 1),\ \alpha_0 = \pi;$$

$$c_i = (\log \alpha_{i-1}/\alpha)^{-1/(1+\eta)}.$$

And define an even function:

$$\phi(t) = c_i \sin \lambda_i t \qquad (\alpha_i < t \leqq \alpha_{i-1};\ i = 1, 2, 3, \cdots).$$

Then, $\phi(t)$, being bounded, is integrable in the Lebesgue sense.

By a number of direct calculations,[3] we can arrive at the following results:

$$(1) \qquad \frac{1}{t} \log \frac{1}{t} \int_0^t \phi(u)du = \frac{1}{\alpha_n} \log \frac{1}{\alpha_n} O\left(\sum_{i \geqq n} \frac{c_i}{\lambda_i}\right) \text{ for } \alpha_n < t \leqq \alpha_{n-1}.$$

Let the λ_nth partial sum of the Fourier series of $\phi(t)$ at $t=0$ be S_{λ_n}, then

$$\begin{aligned} S_{\lambda_n} &= \frac{2}{\pi} \int_0^\pi \phi(t) \frac{\sin \lambda_n t}{t} dt + o(1) \\ &= \frac{2}{\pi}\left(\sum_{i<n} + \sum_{i>n}\right) c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt \\ &\quad + \frac{2c_n}{\pi} \int_{\alpha_n}^{\alpha_{n-1}} \frac{\sin^2 \lambda_n t}{t} dt + o(1) \\ &= \frac{2}{\pi} (\Sigma_1 + \Sigma_2 + \Sigma_3) + o(1). \end{aligned}$$

We have

$$(2) \qquad \begin{aligned} \Sigma_1 &= \sum_{i<n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt \\ &= O\left(\sum_{i<n} \frac{c_i}{\alpha_i(\lambda_n - \lambda_i)}\right) = \frac{1}{\lambda_n} O\left(\sum_{i<n} \frac{c_i}{\alpha_i}\right), \end{aligned}$$

if $\lambda_{n-1}/\lambda_n < k < 1$ for every n.

$$(3) \qquad \Sigma_2 = \sum_{i>n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt = \lambda_n O\left(\sum_{i>n} \frac{c_i \alpha_{i-1}}{\lambda_i \alpha_i}\right),$$

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[2] We take the sequence (λ_i) in such a manner that $\alpha_i < \alpha_{i-1}$ for every i and $c_i = o(1)$ as $i \to \infty$.

[3] For the detailed derivation of these expressions, cf. Proc. London Math. Soc. loc. cit., pp. 316–318.

since the total variation of $\sin \lambda_n t$ in (α_i, α_{i-1}) is $O(\alpha_{i-1}\lambda_n)$.

$$\Sigma_3 = c_n \int_{\alpha_n}^{\alpha_{n-1}} \frac{\sin^2 \lambda_n t}{t} dt = \frac{c_n}{2} \log \frac{\alpha_{n-1}}{\alpha_n} + O\left(\frac{c_n}{\alpha_n \lambda_n}\right). \tag{4}$$

Finally, we have

$$a_n = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos ntdt = \frac{2}{\pi} \left(\sum_{i \leqq \rho_n} + \sum_{i > \rho_n} \right) c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i t \cos ntdt$$
$$= \frac{2}{\pi} \left(\sum_{i \leqq \rho_n} J_i + \sum_{i > \rho_n} J_i \right)$$

say, where (ρ_n) is a rapidly increasing sequence of positive integers.

Now,

$$\sum_{i \leqq \rho_n} J_i = \sum_{i \leqq \rho_n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i t \cos ntdt = \frac{1}{n} O\left(\sum_{i \leqq \rho_n} c_i \lambda_i \alpha_{i-1} \right). \tag{5}$$

On the other hand,

$$\sum_{i > \rho_n} J_i = O\left(\sum_{i > \rho_n} \frac{c_i}{|\lambda_i - n|} \right). \tag{6}$$

In order to establish the theorem, we need to make the value in (1), $o(1)$, $\Sigma_1 + \Sigma_2 = o(1)$, $\Sigma_3 \succ 1$ and the values of the expressions in (5) and (6) are both $o((\log n)^{1+\eta}/n)$.

We take $\lambda_i = [((i+1)!)^\eta]$. Then

$$\frac{1}{t} \log \frac{1}{t} \int_0^t \phi(u) du = ((n+1)!)^\eta o\left(\sum_{i \geqq n} \frac{1}{((i+1)!)^\eta} \right) = o(1)$$

as $t \to +o$. And

$$\Sigma_1 = \frac{1}{((n+1)!)^\eta} O\left(\sum_{i<n} \frac{((i+1)!)^\eta}{\log (i+1)!} \right)$$
$$= \frac{1}{((n+1)!)^\eta} O\left(\frac{(n!)^\eta}{\log n} \right) = o(1).$$

$$\Sigma_2 = ((n+1)!)^\eta o\left(\sum_{i>n} \frac{1}{(i!)^\eta} \right) = o(1).$$

$$\Sigma_3 \cong \frac{\eta}{2(\log (n+1))^{1/(1+\eta)}} \log \frac{(n+1)!}{n!} + o(1)$$
$$= \frac{\eta}{2} (\log (n+1))^{\eta/(1+\eta)} + o(1) \succ 1.$$

This shows that the Fourier series of $\phi(t)$ diverges at $t=0$.

Moreover,

$$\sum_{i\leqq\rho_n} J_i = \frac{1}{n}\, o\left(\sum_{i\leqq\rho_n}((i+1!)^\eta \frac{\log i!}{(i!)^\eta}\right) = \frac{1}{n}\, o(\rho_n^{2+\eta}\log\rho_n),$$

$$\sum_{i>\rho_n} J_i = o\left(\sum_{i>\rho_n}\frac{1}{((i+1)!)^\eta - n}\right) = o\left(\sum_{i>\rho_n}\frac{1}{((i+1)!)^\eta - (i!)^\eta}\right),$$

since $(i!)^\eta \geqq \lambda_{i-1} \geqq \lambda_{\rho_n} > n$. Therefore,

$$\sum_{i>\rho_n} J_i = o\left(\frac{1}{(\rho_n!)^\eta}\right) = o\left(\frac{e^{\eta\rho_n}}{\rho_n^{\eta(\rho_n+1/2)}}\right)$$

by Stirling's formula

$$e^n n! \cong (2\pi)^{1/2} n^{n+1/2}.$$

Now, if we choose a rapidly increasing sequence of positive integers (n_i) such as $n_i = ((i+1)!)!$ and define another function $\phi^*(t)$ from $\phi(t)$ as follows:

$$\phi^*(t) = c_i^* \sin\lambda_i^* t \qquad (\alpha_i^* < t \leqq \alpha_{i-1}^*),$$

where $\lambda_i^* = \lambda_i$, $\alpha_i^* = \alpha_i$ and $c_\nu^* = c_{n_i}$ if $\nu = n_i$ for a certain i; and $c_\nu^* = 0$ if $\nu \neq n_i$ for every i, under this circumstance, the corresponding condition (i) of the theorem is still satisfied and the Fourier series of $\phi^*(t)$ still diverges at $t=0$ as before. But, by this method of choice of the sequence (n_i), we can easily find that the corresponding

$$\sum_{i\leqq\rho_n} J_i^* = \frac{1}{n}\, o(\rho_n^{1+\eta}\log\rho_n).$$

Taking $\rho_n = [\log n(\log^2 n)^{-1/(1+\eta)}]$, we obtain

$$\sum_{i\leqq\rho_n} J_i^* = o\left(\frac{(\log n)^{1+\eta}}{n}\right),$$

also,

$$\sum_{i>\rho_n} J_i^* = o\left(\frac{(\log n)^{1+\eta}}{n}\right).$$

Thus, $a_n^* = o((\log n)^{1+\eta}/n)$. This completes the proof of the theorem.

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