A PROPERTY OF ORDERED RINGS

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In this note we shall provide the last essential element in a simple proof of the theorem of Wagner [1] which states that *every ordered ring satisfying a nontrivial polynomial identity is commutative.*

It is clear that an ordered ring A has no divisors of zero. Moreover, the existence of a nontrivial identity is easily seen to imply that if $a \neq 0$ and $b \neq 0$ are in A there exist elements $c \neq 0$ and $d \neq 0$ in A such that

$$ac = bd. \tag{1}$$

Hence A is what O. Ore [2] has called a *regular* ring. Then A can be imbedded in a unique quotient ring B. Every element of B can be expressed as a product

$$\alpha = b^{-1}a, \tag{2}$$

for $b \neq 0$ and a in A. By (1) we can always write

$$\alpha = b^{-1}a = dc^{-1}. \tag{3}$$

Since $\alpha = (-b)^{-1}(-a) = (-d)(-c)^{-1}$ we can assume that, in the case where A is ordered, the denominators b and c *are always positive.* We now derive the following sequence of simple lemmas.

LEMMA 1. *Let $\alpha = b^{-1}a = dc^{-1}$ where $b > 0$, $c > 0$. Then a and d have the same sign.*

For $b^{-1}a = dc^{-1}$ if and only if $bd = ac$. Since $b > 0$ and $c > 0$ the elements bd and ac of A can be equal only if a and d have the same sign.

LEMMA 2. *Let $\alpha = b^{-1}a = c^{-1}f$ where $b > 0$, $c > 0$. Then a and f have the same sign.*

For we use (3) to write $\alpha = dc^{-1}$ where $c > 0$, and d has the same sign as a. By Lemma 1 we know that f has the same sign as d and hence the same sign as a.

Since the sign of a is unique we may say that $\alpha = b^{-1}a > 0$ if $a > 0$, $\alpha < 0$ if $a < 0$, $\alpha = 0$ if $a = 0$. We may then prove the following result.

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LEMMA 3. *Let $\alpha \neq 0$ be in B and let there exist positive elements a and b of A such that $c = a\alpha b$ is in A. Then c and α have the same sign.*

For $a\alpha = cb^{-1}$ has the same sign as c by our definition of sign. Also $cb^{-1} = e^{-1}f$ where $e > 0$ and f has the same sign as c. Then $\alpha = (ea)^{-1}f$, $ea > 0$, α has the same sign as f and hence the same sign as c.

LEMMA 4. *Let α and β be in B and $\alpha > 0$, $\beta > 0$. Then $\alpha + \beta$ and $\alpha\beta$ are positive.*

For we may write $\alpha = a^{-1}b$, $\beta = dc^{-1}$ with a, b, c, d all positive. Then $a(\alpha+\beta)c = a(a^{-1}b + dc^{-1}) = bc + ad > 0$. By Lemma 3 we have $\alpha + \beta > 0$. Also $a(\alpha\beta)c = (a\alpha)(\beta c) = bd > 0$ and so $\alpha\beta > 0$.

If $\alpha < 0$ and $\beta < 0$ then $-\alpha > 0$, $-\beta > 0$, $(-\alpha)(-\beta) = \alpha\beta > 0$. Similarly if $\alpha < 0$ and $\beta > 0$ we have $-(\alpha\beta) = (-\alpha)\beta > 0$ and $\alpha\beta < 0$. We have completed a proof of the following result.

THEOREM. *The quotient ring of an ordered regular ring is ordered.*

As a consequence of results of Amitsur [**3**] and Kaplansky [**4**] we have the property which states that *if an ordered ring A satisfies a nontrivial polynomial identity the quotient ring B also satisfies the identity and is finite-dimensional over its center F.* By our theorem B is ordered and this order clearly implies that F is ordered. But then it is known [**5**] that B is commutative and so we have Wagner's result that A is commutative.

REFERENCES

1. *Über die Grundlagen der projectiven Geometrie und allgemeine Zahlensysteme*, Math. Ann. vol. 113 (1937) pp. 528–567.

2. *Linear equations in non-commutative fields*, Ann. of Math. vol. 32 (1931) pp. 463–471.

3. *On rings with identities*, J. London Math. Soc. vol. 30 (1955) pp. 464–470.

4. *Rings with a polynomial identity*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 575–580.

5. A. A. Albert, *On ordered algebras*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 521–522.

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