EVALUATION OF A TRIGONOMETRIC INTEGRAL

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1. **Introduction.** It is assumed throughout the present paper that $f(t) \in L(-\infty, \infty)$. Then the Fourier integral of $f(t)$ at $t=x$ is

$$\frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t) \cos u(x-t)dt. \tag{1}$$

This may be written in a form analogous to that of the Fourier series as

$$\int_0^\infty \{a(u) \cos xu + b(u) \sin xu\} du, \tag{2}$$

where

$$a(u) = \frac{1}{\pi}\int_{-\infty}^\infty f(t) \cos utdt, \qquad b(u) = \frac{1}{\pi}\int_{-\infty}^\infty f(t) \sin utdt. \tag{3}$$

The allied integral of the Fourier integral is then

$$\begin{aligned} &\int_0^\infty \{b(u) \cos xu - a(u) \sin xu\} du \\ &\qquad = \frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t) \sin u(t-x)dt. \end{aligned} \tag{4}$$

Thus the allied integral of an odd function $f(t)$ belonging to $L(0, \infty)$ is

$$\frac{2}{\pi}\int_0^\infty \cos xudu \int_0^\infty f(t) \sin utdt. \tag{5}$$

We write

$$\psi(t) = f(x+t) - f(x-t). \tag{6}$$

DEFINITION. We say that the integral $\int_0^\infty g(u)du$ is summable $(C, 1)$ to sum I, if

$$\lim_{\lambda\to\infty} \int_0^\lambda \left(1 - \frac{u}{\lambda}\right) g(u)du = I.$$

In this paper we are concerned with the integrals of the type

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$$\int_0^\infty u^\alpha \{b(u) \cos xu - a(u) \sin xu\} du, \tag{7}$$

where $0<\alpha<1$, and first prove the following

THEOREM. *Integral* (7) *is summable* $(C, 1)$ *to*

$$\frac{\Gamma(1+\alpha)}{\pi} \cos \frac{1}{2} \alpha\pi \int_{\to 0}^\infty \frac{\psi(t)}{t^{1+\alpha}} dt,$$

whenever this integral exists, and

$$\psi(t) = o(t^\alpha) \text{ as } t \to 0. \tag{8}$$

2. **Proof of the theorem.** We are to consider the behaviour of

$$I = \frac{1}{\pi} \int_0^\lambda \left(1 - \frac{u}{\lambda}\right) u^\alpha du \int_{-\infty}^\infty f(t) \sin u(t-x) dt$$

$$= \frac{1}{\pi} \int_0^\lambda \left(1 - \frac{u}{\lambda}\right) u^\alpha du \int_0^\infty \psi(t) \sin ut dt$$

as $\lambda \to \infty$. Now

$$\int_0^\lambda \left(1 - \frac{u}{\lambda}\right) u^\alpha du \int_0^\infty \psi(t) \sin ut dt = \int_0^\infty \psi(t) K(\lambda, t) dt,$$

where

$$K(\lambda, t) = \int_0^\lambda \left(1 - \frac{u}{\lambda}\right) u^\alpha \sin ut du$$

$$= \lambda^{1+\alpha} \int_0^1 (\omega^\alpha - \omega^{\alpha+1}) \sin \omega\lambda t d\omega$$

$$= \lambda^{1+\alpha} \left[(\omega^\alpha - \omega^{\alpha+1}) \frac{\cos \omega\lambda t}{-\lambda t} \right]_0^1$$

$$+ \frac{\lambda^\alpha}{t} \int_0^1 [\alpha\omega^{\alpha-1} - (\alpha+1)\omega^\alpha] \cos \omega\lambda t d\omega$$

$$= \frac{\alpha\lambda^\alpha}{t} \int_0^1 \frac{\cos \omega\lambda t}{\omega^{1-\alpha}} d\omega - \frac{(\alpha+1)\lambda^\alpha}{t} \int_0^1 \omega^\alpha \cos \omega\lambda t d\omega$$

$$= \frac{\alpha}{t^{1+\alpha}} \int_0^{\lambda t} \frac{\cos \theta}{\theta^{1-\alpha}} d\theta + O\left(\frac{1}{\lambda^{1-\alpha} t^2}\right).$$

Hence

$$K(\lambda, t) - \frac{\alpha}{t^{1+\alpha}} \int_0^\infty \frac{\cos\theta}{\theta^{1-\alpha}} d\theta = \frac{-\alpha}{t^{1+\alpha}} \int_{\lambda t}^\infty \frac{\cos\theta}{\theta^{1-\alpha}} d\theta + O\left(\frac{1}{\lambda^{1-\alpha} t^2}\right).$$

By the second mean value theorem, we have

$$\left| \int_{\lambda t}^\infty \frac{\cos\theta}{\theta^{1-\alpha}} d\theta \right| \leqq \frac{2}{\lambda^{1-\alpha} t^{1-\alpha}},$$

and consequently

$$\left| K(\lambda, t) - \frac{\Gamma(\alpha+1)}{t^{1+\alpha}} \cos\frac{1}{2}\alpha\pi \right| < \frac{A}{\lambda^{1-\alpha} t^2}. \tag{9}$$

Plainly

$$| K(\lambda, t) | < A\lambda^{1+\alpha}. \tag{10}$$

It follows at once that, if $\delta > 0$,

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_\delta^\infty \psi(t) K(\lambda, t) dt = \frac{\Gamma(1+\alpha)}{\pi} \cos\frac{1}{2}\alpha\pi \int_\delta^\infty \frac{\psi(t)}{t^{1+\alpha}} dt.$$

Also by (9), we have

$$\frac{1}{\pi} \int_{\lambda^{-1}}^\delta \psi(t) K(\lambda, t) dt$$
$$= \frac{\Gamma(1+\alpha)}{\pi} \cos\frac{1}{2}\alpha\pi \int_{\lambda^{-1}}^\delta \frac{\psi(t)}{t^{1+\alpha}} dt + O\left\{ \int_{\lambda^{-1}}^\delta \frac{|\psi(t)|}{t^2 \lambda^{1-\alpha}} dt \right\}$$

and, if condition (8) holds, the last term is

$$o\left(\frac{1}{\lambda^{1-\alpha}} \int_{\lambda^{-1}}^\delta t^{\alpha-2} dt \right) = o(1)$$

by choosing first δ then λ.

Finally, by (8) and (10), we have

$$\left| \frac{1}{\pi} \int_0^{\lambda^{-1}} \psi(t) K(\lambda, t) dt \right| < A\lambda^{1+\alpha} \int_0^{\lambda^{-1}} | \psi(t) | dt$$
$$= o(1).$$

This proves the theorem.

3. **Evaluation of a definite integral.** Let us consider the odd function $e^{-t} t^{\beta-1}$ $(\beta > 1+\alpha)$ defined in $(0, \infty)$, then the integral (7) at $x=0$ reduces for the present function to

$$\frac{2}{\pi}\int_0^\infty u^\alpha du \int_0^\infty e^{-t}t^{\beta-1}\cdot \sin utdt$$

$$= \frac{2}{\pi}\Gamma(\beta)\int_0^\infty \frac{u^\alpha \sin\left\{\beta \tan^{-1} u\right\}du}{(1+u^2)^{\beta/2}}, \quad \text{where} \quad \beta > 1+\alpha.$$

Obviously this integral is a convergent integral, and since this special function satisfies the conditions of the theorem, we have by the theorem

$$(11) \qquad \int_0^\infty \frac{u^\alpha \sin(\beta \tan^{-1} u)}{(u^2+1)^{\beta/2}}\, du = \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}{\Gamma(\beta)} \cos\frac{1}{2}\alpha\pi,$$

where

$$\beta > \alpha + 1.$$

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