EVALUATION OF A TRIGONOMETRIC INTEGRAL

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1. Introduction. It is assumed throughout the present paper that \( f(t) \in L(-\infty, \infty) \). Then the Fourier integral of \( f(t) \) at \( t=x \) is

\[
\frac{1}{\pi} \int_{0}^{\infty} du \int_{-\infty}^{\infty} f(t) \cos u(x-t) dt.
\]

This may be written in a form analogous to that of the Fourier series as

\[
\int_{0}^{\infty} \{ a(u) \cos xu + b(u) \sin xu \} du,
\]

where

\[
a(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos ut dt, \quad b(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin ut dt.
\]

The allied integral of the Fourier integral is then

\[
\int_{0}^{\infty} \{ b(u) \cos xu - a(u) \sin xu \} du
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} du \int_{-\infty}^{\infty} f(t) \sin u(t-x) dt.
\]

Thus the allied integral of an odd function \( f(t) \) belonging to \( L(0, \infty) \) is

\[
\frac{2}{\pi} \int_{0}^{\infty} \cos xu \int_{0}^{\infty} f(t) \sin ut dt.
\]

We write

\[
\psi(t) = f(x+t) - f(x-t).
\]

**Definition.** We say that the integral \( \int_{0}^{\infty} g(u) du \) is summable \((C, 1)\) to sum \( I \), if

\[
\lim_{\lambda \to \infty} \int_{0}^{\lambda} \left( 1 - \frac{u}{\lambda} \right) g(u) du = I.
\]

In this paper we are concerned with the integrals of the type

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(7) \[ \int_{0}^{\infty} u^\alpha \{ b(u) \cos xu - a(u) \sin xu \} \, du, \]

where \( 0 \leq \alpha \leq 1 \), and first prove the following

**Theorem.** Integral (7) is summable \((C, 1)\) to

\[
\frac{\Gamma(1 + \alpha)}{\pi} \cos \frac{1}{2} \alpha \pi \int_{-\infty}^{\infty} \frac{\psi(t)}{t^{1+\alpha}} \, dt,
\]

whenever this integral exists, and

(8) \[ \psi(t) = o(t^\alpha) \text{ as } t \to 0. \]

2. **Proof of the theorem.** We are to consider the behaviour of

\[
I = \frac{1}{\pi} \int_{0}^{\lambda} \left(1 - \frac{u}{\lambda}\right) u^\alpha \sin u(t - x) \, du \int_{-\infty}^{\infty} f(t) \sin u(t - x) \, dt
\]

\[
= \frac{1}{\pi} \int_{0}^{\lambda} \left(1 - \frac{u}{\lambda}\right) u^\alpha \, du \int_{0}^{\infty} \psi(t) \sin ut \, dt
\]

as \( \lambda \to \infty \). Now

\[
\int_{0}^{\lambda} \left(1 - \frac{u}{\lambda}\right) u^\alpha \, du \int_{0}^{\infty} \psi(t) \sin ut \, dt = \int_{0}^{\infty} \psi(t) K(\lambda, t) \, dt,
\]

where

\[
K(\lambda, t) = \int_{0}^{\lambda} \left(1 - \frac{u}{\lambda}\right) u^\alpha \sin u t \, du
\]

\[
= \lambda^{1+\alpha} \int_{0}^{1} (\omega - \omega^{\alpha+1}) \sin \omega \lambda t d\omega
\]

\[
= \lambda^{1+\alpha} \left[ (\omega - \omega^{\alpha+1}) \frac{\cos \omega \lambda t}{-\lambda t} \right]_{0}^{1}
\]

\[
+ \frac{\lambda^\alpha}{t} \int_{0}^{1} [\alpha \omega^{\alpha-1} - (\alpha + 1) \omega^\alpha] \cos \omega \lambda t d\omega
\]

\[
= \frac{\alpha \lambda^\alpha}{t} \int_{0}^{1} \frac{\cos \omega \lambda t}{\omega^{1-\alpha}} \, d\omega - \frac{(\alpha + 1) \lambda^\alpha}{t} \int_{0}^{1} \omega^\alpha \cos \omega \lambda t d\omega
\]

\[
= \frac{\alpha}{t^{1+\alpha}} \int_{0}^{\lambda} \cos \theta \frac{1}{\theta^{1-\alpha}} \, d\theta + O \left( \frac{1}{\lambda^{1+\alpha}} \right).
\]

Hence
By the second mean value theorem, we have

\[
\left| \int_{\lambda t}^{\infty} \frac{\cos \theta}{\theta^{1-\alpha}} \, d\theta \right| \leq \frac{2}{\lambda^{1-\alpha} t^{1-\alpha}},
\]

and consequently

\[
\left| K(\lambda, t) - \frac{\Gamma(\alpha + 1)}{t^{1+\alpha}} \cos \frac{1}{2} \alpha \pi \right| < \frac{A}{\lambda^{1-\alpha} t^2}.
\]

Plainly

\[
| K(\lambda, t) | < A \lambda^{1+\alpha}.
\]

It follows at once that, if \( \delta > 0 \),

\[
\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{\delta}^{\infty} \psi(t) K(\lambda, t) \, dt = \frac{\Gamma(1+\alpha)}{\pi} \cos \frac{1}{2} \alpha \pi \int_{\delta}^{\infty} \frac{\psi(t)}{t^{1+\alpha}} \, dt.
\]

Also by (9), we have

\[
\frac{1}{\pi} \int_{\lambda}^{\delta} \psi(t) K(\lambda, t) \, dt
= \frac{\Gamma(1+\alpha)}{\pi} \cos \frac{1}{2} \alpha \pi \int_{\lambda}^{\delta} \frac{\psi(t)}{t^{1+\alpha}} \, dt + O \left\{ \int_{\lambda}^{\delta} \frac{|\psi(t)|}{t^{1+\alpha} \lambda^{1-\alpha}} \, dt \right\}
\]

and, if condition (8) holds, the last term is

\[
o \left( \frac{1}{\lambda^{1-\alpha}} \int_{\lambda}^{\delta} t^{-\alpha} \, dt \right) = o(1)
\]

by choosing first \( \delta \) then \( \lambda \).

Finally, by (8) and (10), we have

\[
\left| \frac{1}{\pi} \int_{0}^{\lambda^{-1}} \psi(t) K(\lambda, t) \, dt \right| < A \lambda^{1+\alpha} \int_{0}^{\lambda^{-1}} |\psi(t)| \, dt = o(1).
\]

This proves the theorem.

3. **Evaluation of a definite integral.** Let us consider the odd function \( e^{-t^{\beta-1}} (\beta > 1+\alpha) \) defined in \((0, \infty)\), then the integral (7) at \( x = 0 \) reduces for the present function to
\[
\frac{2}{\pi} \int_0^\infty u^\alpha du \int_0^\infty e^{-t^{\beta-1}} \sin u \, dt
= \frac{2}{\pi} \Gamma(\beta) \int_0^\infty \frac{u^\alpha \sin \{\beta \tan^{-1} u\} \, du}{(1 + u^2)^{\beta/2}}, \quad \text{where} \quad \beta > 1 + \alpha.
\]

Obviously this integral is a convergent integral, and since this special function satisfies the conditions of the theorem, we have by the theorem

\[
(11) \quad \int_0^\infty \frac{u^\alpha \sin (\beta \tan^{-1} u)}{(u^2 + 1)^{\beta/2}} \, du = \frac{\Gamma(\alpha + 1)\Gamma(\beta - \alpha - 1)}{\Gamma(\beta)} \cos \frac{1}{2} \alpha \pi,
\]

where

\[
\beta > \alpha + 1.
\]

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