DERIVATIONS OF NILPOTENT LIE ALGEBRAS

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In a recent note Jacobson proved [1] that, over a field of characteristic 0, a Lie algebra with a nonsingular derivation is nilpotent. He also noted that the validity of the converse was an open question. The purpose of this note is to supply a strongly negative answer to that question and to point out some of the immediate problems which this answer raises.

Suppose then that Φ is a field of characteristic 0 and that $\mathfrak{L}$ is the 8 dimensional algebra over Φ described in terms of a basis $e_1, e_2, \cdots, e_8$ by the following multiplication table:

(1)	$[e_1, e_2] = e_5,$	(6)	$[e_2, e_4] = e_6,$
(2)	$[e_1, e_3] = e_6,$	(7)	$[e_2, e_6] = -e_7,$
(3)	$[e_1, e_4] = e_7,$	(8)	$[e_3, e_4] = -e_5,$
(4)	$[e_1, e_5] = -e_8,$	(9)	$[e_3, e_5] = -e_7,$
(5)	$[e_2, e_3] = e_8,$	(10)	$[e_4, e_6] = -e_8.$

In addition $[e_i, e_j] = -[e_j, e_i]$ and for $i<j\,[e_i, e_j]=0$ if it is not in the table above. Note that all triple products $[[e_ie_j]e_k]$ vanish if one index is >4. It is convenient to use a symmetry in the table above. Denote by A the linear transformation induced in $\mathfrak{L}$ by the mapping

$$\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ e_3 & e_4 & e_1 & e_2 & -e_5 & -e_6 & -e_8 & -e_7 \end{pmatrix}.$$

A direct check shows that A is an automorphism of $\mathfrak{L}$. By observing

$$[[e_1\ e_2]e_3] + [[e_2\ e_3]e_1] + [[e_3\ e_1]e_2] = e_7 - e_7 = 0$$

and

$$[[e_1\ e_2]e_4] + [[e_2\ e_4]e_1] + [[e_4\ e_1]e_2] = 0,$$

and by applying A to each we conclude that $\mathfrak{L}$ is a Lie algebra.

Since $\mathfrak{L}^2 = \{e_5, e_6, e_7, e_8\}$, $\mathfrak{L}^3 = \{e_7, e_8\}$, $\mathfrak{L}^4 = \{0\}$, $\mathfrak{L}$ is nilpotent.

THEOREM. *If D is a derivation of $\mathfrak{L}$ then $\mathfrak{L}D \subset \mathfrak{L}^2$; hence every derivation is nilpotent.*

PROOF. Suppose $e_iD = \sum \delta_{ij}e_j$, $1 \leqq i \leqq 8$, $1 \leqq j \leqq 8$. The equations

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$$[e_1\ e_2]D = e_5D = \delta_{55}e_5 + \delta_{56}e_6 + \delta_{57}e_7 + \delta_{58}e_8,$$
$$[e_1D,\ e_2] = \delta_{11}e_5 - \delta_{14}e_6 + \delta_{16}e_7 - \delta_{13}e_8,$$
$$[e_1,\ e_2D] = \delta_{22}e_5 + \delta_{23}e_6 + \delta_{24}e_7 - \delta_{25}e_8,$$

imply

$$\delta_{55} = \delta_{11} + \delta_{22},\quad \delta_{56} = \delta_{23} - \delta_{14},\quad \delta_{57} = \delta_{16} + \delta_{24},\quad \delta_{58} = -\delta_{13} - \delta_{25}.$$

With the observation that for $i \leqq 4$ and $5 \leqq k \leqq 8$ there is exactly one $j \leqq 6$ for which $[e_i, e_j] = \pm e_k$ it follows from (2) that

$$\delta_{65} = \delta_{14} + \delta_{32},\quad \delta_{66} = \delta_{11} + \delta_{33},\quad \delta_{67} = \delta_{15} + \delta_{34},\quad \delta_{68} = \delta_{12} - \delta_{35},$$

from (3) that

$$\delta_{77} = \delta_{11} + \delta_{44},\quad \delta_{78} = \delta_{16} - \delta_{45},\quad \delta_{13} = \delta_{42},\quad \delta_{12} = -\delta_{43}$$

and from (4) that

$$\delta_{87} = \delta_{13},\qquad \delta_{88} = \delta_{11} + \delta_{55}.$$

The automorphism A transforms D into a derivation D^* by $A^{-1}DA = D^*$ and this implies that with each equation $\delta_{ij} = \delta_{kl} + \delta_{mn}$ there is also valid $\delta_{\alpha(i),\alpha(j)} = \delta_{\alpha(k),\alpha(l)} + \delta_{\alpha(m),\alpha(n)}$ where α is the permutation of $\{-8, -7, \cdots, 7, 8\}$ induced by A and where $\delta_{(-1)^p i,(-1)^q j} = (-1)^{p+q}\delta_{ij}$. D operating on (6) provides

$$\delta_{65} = -\delta_{23} - \delta_{41},\quad \delta_{66} = \delta_{22} + \delta_{44},\quad \delta_{67} = \delta_{21} - \delta_{46},\quad \delta_{68} = \delta_{26} + \delta_{43},$$

and (7) gives

$$\delta_{77} = \delta_{66} + \delta_{22},\qquad \delta_{78} = \delta_{24}.$$

From the vanishing of $[e_1, e_6]$ follows $\delta_{12} = 0$, $\delta_{14} = -\delta_{65}$ and from $[e_2, e_5] = 0$, $\delta_{21} = 0$, $\delta_{23} = -\delta_{56}$. Again, another set of equations is obtained by applying A.

Among the ten relations of the form $\delta_{ii} + \delta_{jj} = \delta_{kk}$, eight are linearly independent so that $\delta_{ii} = 0$ for $i = 1, 2, \cdots, 8$. The relations

$$\delta_{78} = \delta_{31} = \delta_{24} = \delta_{16} - \delta_{45}$$

and

$$\delta_{57} = \delta_{45} - \delta_{31} = \delta_{16} + \delta_{24}$$

imply

$$\delta_{57} = \delta_{16} = \delta_{45} \quad\text{and}\quad \delta_{24} = \delta_{78} = \delta_{31} = 0,$$

and there are also

$$\delta_{58} = -\delta_{36} = -\delta_{25}, \qquad \delta_{42} = \delta_{87} = \delta_{13} = 0.$$

$\delta_{21}=0$ implies $\delta_{43}=0$. The relations $\delta_{68}=\delta_{12}-\delta_{35}=\delta_{26}+\delta_{43}$ imply $\delta_{68}=-\delta_{35}=\delta_{26}$, and thus also $\delta_{67}=\delta_{15}=-\delta_{46}$. Since $\delta_{23}=-\delta_{56}$ and $-\delta_{65}=\delta_{23}+\delta_{41}=-\delta_{14}-\delta_{32}=\delta_{41}=\delta_{14}$,

$$\delta_{65} = \delta_{41} = \delta_{23} = \delta_{14} = \delta_{32} = \delta_{56} = 0.$$

The matrix of D is therefore:

$$\left[\begin{array}{c|cccc}
 & \delta_{67} & \delta_{57} & \delta_{17} & \delta_{18} \\
0 & -\delta_{58} & \delta_{68} & \delta_{27} & \delta_{28} \\
 & -\delta_{68} & -\delta_{58} & \delta_{37} & \delta_{38} \\
 & \delta_{57} & -\delta_{67} & \delta_{47} & \delta_{48} \\
\hline
 & \multicolumn{2}{c|}{0} & \delta_{57} & \delta_{58} \\
0 & \multicolumn{2}{c|}{} & \delta_{67} & \delta_{68} \\
\cline{4-5}
 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0}
\end{array}\right]$$

The derivation algebra $\mathfrak{D}$ is 12 dimensional and the algebra $\mathfrak{J}$ of inner derivations 6 dimensional. Every linear transformation sending $\mathfrak{L}$ into the center of $\mathfrak{L}$ and $\mathfrak{L}^2$ into 0 is a derivation. The ideal $\mathfrak{D}_0$ of these derivations is 8 dimensional and intersects $\mathfrak{J}$ in a 2 dimensional space. In a sense then $\mathfrak{L}$ has as few outer derivations as possible. More precisely,

$$\mathfrak{N} = \mathfrak{D}_0 + \mathfrak{J}.$$

The existence of $\mathfrak{L}$ suggests the consideration of a subclass of nilpotent algebras which might prove more tractable than the entire class. To this end, for any algebra $\mathfrak{N}$ with derivation algebra $\mathfrak{D}$, let

$$\mathfrak{N}^{[1]} = \mathfrak{N}\mathfrak{D} = \left\{ \sum x_i D_i \,\middle|\, x_i \in \mathfrak{N},\, D_i \in \mathfrak{D} \right\},$$

and let

$$\mathfrak{N}^{[k+1]} = \mathfrak{N}^{[k]}\mathfrak{D}.$$

$\mathfrak{N}$ could be called *characteristically nilpotent* if for some k, $\mathfrak{N}^{[k]}=0$. The algebra $\mathfrak{L}$ is characteristically nilpotent and for any such algebra $\mathfrak{N}$,

(1) if $\mathfrak{N}$ is an ideal of a solvable algebra $\mathfrak{N}$ then either $\mathfrak{N}^k \not\subset \mathfrak{N}$ for any k or $\mathfrak{N}$ is nilpotent,

(2) if $\mathfrak{M}$ is an algebra with nil-radical $\mathfrak{N}$ then $\mathfrak{M}=\mathfrak{N}\oplus\mathfrak{S}$ for some semi-simple ideal $\mathfrak{S}$.

One might ask whether there is an intrinsic characterization of such algebras, and a general method for constructing them all.

The algebra $\mathfrak{L}$ has an additional property which may be shared by all characteristically nilpotent algebras: $\mathfrak{L}$ is not the derived algebra of any Lie algebra. To see this, observe that $\mathfrak{L}^{[1]}=\mathfrak{L}\mathfrak{D}=\mathfrak{L}^2$ so that if $\mathfrak{L}\subset\mathfrak{M}$ and $\mathfrak{M}^2=\mathfrak{L}$, then $[\mathfrak{L}\mathfrak{M}]\subset\mathfrak{L}^{[1]}=\mathfrak{L}^2$. This implies $\mathfrak{L}^2=[\mathfrak{L}[\mathfrak{M}\mathfrak{M}]]\subset[[\mathfrak{L}\mathfrak{M}]\mathfrak{M}]\subset[\mathfrak{L}^2\mathfrak{M}]\subset[\mathfrak{L}^2\mathfrak{L}]$, and this contradicts the nilpotency of $\mathfrak{L}$.

Reference

1. N. Jacobson, *A note on automorphisms and derivations of Lie algebras*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 281–283.

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