REGULAR COLLINEATION GROUPS

D. R. HUGHES[1]

1. **Introduction.** Let v, k, λ, $(v>k>\lambda>0)$ be integers satisfying $\lambda(v-1)=k(k-1)$. Suppose π is a collection of v points and v lines, together with an incidence relation such that every point (line) is on k lines (contains k points), and such that every pair of distinct points (lines) are on λ common lines (contain λ points in common). Then π is a λ-plane, or a (v, k, λ) configuration, or a symmetric balanced incomplete block design (see [**1**; **3**] for more details). If ϕ is a one-to-one mapping of π, sending points onto points and lines onto lines, and preserving incidence, then ϕ is a *collineation* of π. If π is a λ-plane possessing a collineation group $\mathfrak{G}$ of order m such that no nonidentity element of $\mathfrak{G}$ fixes any point or line of π, then we say that π is *regular of degree m* (with group $\mathfrak{G}$). Any λ-plane is regular of degree one, and the "transitive λ-planes" of [**1**] (including the "cyclic λ-planes" of [**4**; **5**]) are regular of degree v (which is clearly the maximum degree of regularity). In this paper we show that regularity implies the existence of a matrix relation similar to the well-known relations involving incidence matrices (see [**2**; **3**]), and indeed, includes these incidence matrix relations as special cases. If $\lambda=1$, then π is a finite projective plane of order $n=k-1$, and we shall be particularly interested in the fact that the theorems of this paper are strong enough to prove, for a wide class of integers n, that no projective plane of order n can be regular of degree greater than one.

2. **Regular λ-planes.** Let π be a λ-plane with parameters v, k, λ, and suppose π is regular of degree m, with group $\mathfrak{G}$. Then the v points of π break up into t classes $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_t$, each containing m points, such that $\mathfrak{G}$ is transitive (and regular) on any $\mathcal{P}_i$; similarly, the lines break up into t classes $\mathcal{J}_1, \mathcal{J}_2, \cdots, \mathcal{J}_t$, on each of which $\mathfrak{G}$ is transitive. Clearly $mt=v$. In each $\mathcal{P}_i$ choose a "base point" P_i, and in each $\mathcal{J}_i$ a "base line" J_i. Then every point of $\mathcal{P}_i$ can be expressed uniquely in the form P_ix, $x\in\mathfrak{G}$, and every line of $\mathcal{J}_i$ can be expressed uniquely in the form J_ix, $x\in\mathfrak{G}$.

Let D_{ij} be the subset of $\mathfrak{G}$ consisting of all elements x such that P_ix is on J_j, and let n_{ij} be the number of elements in D_{ij}; then $n_{ij}\geqq 0$.

Presented to the Society, April 14, 1956; received by the editors February 20, 1956.

[1] This research was carried out while the author was a National Science Foundation Fellow.

THEOREM 1. (i) *For each* i $(i=1, 2, \cdots, t)$ *and each* $a\in\mathfrak{G}$, $a\neq 1$, *there are exactly* λ *equations of the form* $a=d_1d_2^{-1}$, *where* d_1, d_2 *are in some* D_{ij}; *similarly, there are exactly* λ *equations of the form* $a=d_1^{-1}d_2$, *where* d_1, d_2 *are in some* D_{ji}.

(ii) *For each pair* $i, j, i\neq j$ $(i, j=1, 2, \cdots, t)$, *and each* $a\in\mathfrak{G}$, *there are exactly* λ *equations of the form* $a=d_1d_2^{-1}$, *where* d_1 *is in* D_{jl}, d_2 *is in* D_{il}, *for some* l; *similarly, there are exactly* λ *equations of the form* $a=d_1^{-1}d_2$, *where* d_1 *is in* D_{lj}, d_2 *is in* D_{li}, *for some* l.

PROOF. Given i and $a\in\mathfrak{G}$, $a\neq 1$, consider the λ lines joining P_i and P_ia; there must be exactly λ values of j and b such that P_i, P_ia are on J_jb. For each such line J_jb, we have $ab^{-1}=d_1\in D_{ij}$, $b^{-1}=d_2\in D_{ij}$, whence $a=d_1d_2^{-1}$. By a reversal of the argument, it is easy to see that these λ equations are unique.

Given $i, j, i\neq j$, and $a\in\mathfrak{G}$, consider the λ lines joining P_i and P_ja; there must be exactly λ values of l and b such that P_i, P_ja are on J_lb. Then, as above, we get λ equations $a=d_1d_2^{-1}$, where d_1 is in D_{jl}, d_2 is in D_{il}, and these λ equations are unique.

By similar considerations with the λ points P_lb on J_i and J_ia, or on J_i and J_ja, the other halves of (i) and (ii) are proven.

THEOREM 2. *Letting* $n=k-\lambda$, *the* n_{ij} *satisfy*:

(i) $\sum_j n_{ij}=\sum_j n_{ji}=k$, *for any* i.

(ii) $\sum_j n_{ij}^2=\sum_j n_{ji}^2=n+\lambda m$, *for any* i.

(iii) $\sum_l n_{il}n_{jl}=\sum_l n_{li}n_{lj}=\lambda m$, *for any* $i, j, i\neq j$.

PROOF. The line J_i contains n_{ji} points of $\mathcal{P}_j$, hence contains altogether $\sum_j n_{ji}=k$ points. Since n_{ij} images of P_i are on J_j, there are n_{ij} images of J_j containing P_i, hence n_{ij} lines of $\mathcal{J}_j$ through P_i. So P_i is on altogether $k=\sum_j n_{ij}$ lines. Thus we have (i).

For a fixed i, each $a\in\mathfrak{G}$, $a\neq 1$, is represented exactly λ times among all the elements $d_1d_2^{-1}$, d_1, $d_2\in D_{ij}$, as j varies. On the other hand, all the elements $d_1d_2^{-1}$, d_1, $d_2\in D_{ij}$, $d_1\neq d_2$, as j varies, make up a set of $\sum_j n_{ij}(n_{ij}-1)$ elements. Hence $\sum_j n_{ij}(n_{ij}-1)=\lambda(m-1)$, or $\sum_j n_{ij}^2=\lambda m-\lambda+k=n+\lambda m$, using (i). The other half of (ii) is similar.

Finally, for a fixed $i, j, i\neq j$, each $a\in\mathfrak{G}$ is represented exactly λ times as $a=d_1d_2^{-1}$, where $d_1\in D_{jl}$, $d_2\in D_{il}$, as l varies. Thus $\sum_l n_{il}n_{jl}$ must count every element of $\mathfrak{G}$ λ times, so $\sum_l n_{il}n_{jl}=\lambda m$. The other half of (iii) is similar.

Now if A is a matrix, let A^T be the transpose of A. Then it is im mediate that Theorem 2 can be rephrased as follows, where $A=(n_{ij})$.

THEOREM 3. *If* π *is a regular* λ-*plane of degree* m, *with parameters* v, k, λ, *then there is a square matrix* A *of order* $t=v/m$, *consisting en-*

tirely of non-negative integral entries, such that $A^TA = AA^T = B$, where B has $n+\lambda m$ on the main diagonal and λm elsewhere. Furthermore, every row or column of A sums to k. (Here $n = k-\lambda$.)

THEOREM 4. *If π is a regular λ-plane of degree m, with the parameters v, k, λ, and if $t = v/m$ is odd, then the equation*

$$x^2 = ny^2 + (-1)^{(t-1)/2}\lambda m z^2, \qquad (1)$$

where $n = k-\lambda$, possesses a nontrivial solution in integers.

PROOF. The theorem follows from Theorem 3 and the Lemma of [5].

The equation (1) can be handled by the classical theory of Legendre, and yields nontrivial information for many choices of v, k, λ. If $m=1$, then the matrix relation of Theorem 3 is exactly the incidence matrix equation for a λ-plane [2; 3]; in that light, the concept of a regular λ-plane can be thought of as a notion which includes the most basic combinatorial (or geometric) information as a "special case."

For $\lambda > 1$, $k \leqq 30$, it is fairly easy to investigate all λ-planes, in connection with Theorem 4. If v is a prime, then either $m=1$ or $m=v$, so we disregard these cases; furthermore we neglect those choices of v, k, λ, which are rejected by [3] (i.e., by Theorem 4 with $m=1$). Of the remaining cases, the following cannot be regular of degree greater than one (the parameters (v, k, λ) are listed): (25, 9, 3), (25, 16, 10), (121, 16, 2), (39, 19, 9), (39, 20, 10), (201, 25, 3), (55, 27, 13), (55, 28, 14). Although Theorem 4 gives no direct information about $m=v$, if m is rejected for some prime-power divisor of v, then $m=v$ is impossible: for a regular group of order $v=p^aq$, p a prime, $a>0$, certainly contains a regular subgroup of order p^a. Thus (245, 27, 3) is handled: for $m=7$ or $m=49$, Theorem 4 offers no information, but $m=5$ is impossible, and so $m=245$ is also impossible. (In this connection see Theorem 4.1 of [1] and Theorem 2.1 of [4].)[2]

3. **Regular projective planes.** If $\lambda=1$, then π is a finite projective plane of order $n=k-1$, and $v=n^2+n+1$. Since v is always odd, $t=v/m$ is always odd, so Theorem 4 always applies.

There are 18 integers $n \leqq 60$ which are not prime-powers, are not rejected by [2], and for which n^2+n+1 is not a prime. For eight of these integers (18, 28, 36, 44, 45, 48, 52, 56), Theorem 4 gives no information. For eight others (10, 26, 34, 35, 39, 40, 51, 60), Theorem

[2] In papers forthcoming in the Transactions of the American Mathematical Society and the Illinois Journal of Mathematics the author gives a more general treatment of collineations of (v, k, λ) configurations, including nonexistence theorems.

4 tells us that any plane of order n cannot be regular of degree $m>1$. For $n=55$, $n^2+n+1=3\cdot 13\cdot 79$, Theorem 4 rejects all values of $m>1$ except $m=79$ or $m=39$; but since $m=3$ and $m=13$ are rejected, $m=39$ is impossible. (Thus, oddly, Theorem 4 can give more information indirectly than directly.) For $n=58$, $n^2+n+1=3\cdot 7\cdot 163$, Theorem 4 rejects all values of $m>1$ except $m=7$. In each of these cases, some prime divisor of n^2+n+1 is rejected, so $m=n^2+n+1$ is impossible.

Any Desarguesian projective plane of order n is regular of degree m, for any m dividing n^2+n+1 (see [7]). The only other examples (known to the author) of planes which are regular of degree $m>1$ are the planes given in [6]: the planes of this class are all non-Desarguesian and a typical plane has order p^{2a}, p an odd prime, and is regular of degree m for any m dividing $p^{2a}+p^a+1$.

Bibliography

1. R. H. Bruck, *Difference sets in a finite group*, Trans. Amer. Math. Soc. vol. 78 (1955) pp. 464–481.

2. R. H. Bruck and H. J. Ryser, *The non-existence of certain finite projective planes*, Canadian Journal of Mathematics vol. 1 (1949) pp. 88–93.

3. S. Chowla and H. J. Ryser, *Combinatorial problems*, Canadian Journal of Mathematics vol. 2 (1950) pp. 93–99.

4. Marshall Hall, Jr., *Cyclic projective planes*, Duke Math. J. vol. 14 (1947) pp. 1079–1090.

5. Marshall Hall, Jr. and H. J. Ryser, *Cyclic incidence matrices*, Canadian Journal of Mathematics vol. 3 (1951) pp. 495–502.

6. D. R. Hughes, *A class of non-Desarguesian projective planes*, To appear in Canadian Journal of Mathematics.

7. James Singer, *A theorem in finite projective geometry and some applications to number theory*, Trans. Amer. Math. Soc. vol. 43 (1938) pp. 377–385.

The Ohio State University