NOTE ON SOME GAP THEOREMS

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Many theorems in the theory of series concern gap series of the form

$$(1)\qquad \sum_{m=0}^{\infty} a_m \qquad \begin{array}{r} \text{with } a_m = 0 \text{ for } m_k < m \leqq M_k \text{ and } m_k \nearrow \infty, \\ \text{where } M_k \geqq m_k(1+\vartheta) \text{ for some } \vartheta > 0, \end{array}$$

with partial sums s_m. These theorems infer the convergence of the sequence $\{s_{m_k}\}$ for $k \to \infty$ from assumptions concerning the summability of (1) or from properties of the associated complex function

$$(2)\qquad f(z) = \sum_{m=0}^{\infty} a_m z^m \qquad (z = x + iy).$$

Two theorems of the latter type are the following Theorems A and B.[1]

THEOREM A [1]. *Suppose that a series* (1) *is given and that* (2) *is regular in* $|z| < 1$ *and continuous in a circle* $|z-\alpha| \leqq 1-\alpha$ *for some* α *with* $0<\alpha<1$. *Then* $s_{m_k} \to f(1)$ $(k \to \infty)$.[2]

THEOREM B [4]. *Suppose that a series* (1) *is given and that* (2) *is regular in* $|z| < 1$ *and bounded in a sector* $|\arg z| < \epsilon$, $0 < |z| < 1$. *Then* $\lim_{x \to 1-0} f(x) = s$ *implies* $s_{m_k} \to s$ $(k \to \infty)$.

We are going to prove a theorem which contains both of these theorems:

THEOREM 1. *Suppose that a series* (1) *is given and that* (2) *is regular in* $|z| < 1$ *and bounded in a circle* $|z-\alpha| < 1-\alpha$ *for some* α *with* $0<\alpha<1$. *Then* $\lim_{x \to 1-0} f(x) = s$ *implies* $s_{m_k} \to s$ $(k \to \infty)$.

By a short complex variable argument, we shall reduce the proof of Theorem 1 to the following gap theorem on summable series of type (1).

THEOREM 2. *Suppose that a series* (1) *is given and that*

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[1] In this connection, see also some gap theorems by Noble [8; 9].

[2] Evgrafov has also given an example to show that Theorem A may be false if $|z-\alpha| \leqq 1-\alpha$ is replaced by a sector $\pi/2+\delta \leqq \arg (z-1) \leqq 3\pi/2-\delta$, $0 \leqq |z-1| \leqq \rho$ $(\delta>0,\ \rho>0)$.

(i) *the function* (2) *is regular in* $|z| < \alpha$ *and at* $z = \alpha$ *for some* α *with* $0 < \alpha < 1$;

(ii) *at the point* $z = 1$, *the Taylor series of* $f(z)$ *about* $z = \alpha$ *is* C_1-*summable to the value* s.

Then there exists a number $\delta = \delta(\vartheta, \alpha) > 0$ *such that under the additional hypothesis*

$$\text{(iii)} \qquad s_m = O((1+\delta)^m) \qquad (m \to \infty)$$

one can conclude

$$(3) \qquad s_{m_k} \to s \qquad (k \to \infty).$$

REMARKS. (a) An equivalent form of (ii) is

(ii′) *the series* (1) *is* C_1T_α-*summable to the value* s, *where* T_α *denotes the "circle method" of order* α.[3]

(b) Our proof of Theorem 2 goes beyond the theorem; it establishes the following extension:

The conclusion (3) *in Theorem* 2 *remains valid if* C_1-*summability in* (ii) *or* (ii′) *is replaced by* C_κ-*summability*; *here* κ *can be any number* $\geqq 0$.

For $\kappa = 0$ a similar theorem, with B (Borel) instead of T_α, has been proved by Zygmund (see [5, p. 206]); our theorem or its extension can, like Zygmund's theorem, be used for a proof of Ostrowski's theorem on overconvergence. Theorems similar to Theorem 2 and its extension could also be obtained by our method of proof for B (Borel) and E_p (Euler-Knopp) instead of T_α.

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PROOF OF THEOREM 1. We show that all conditions of Theorem 2 are satisfied. Consider the development $\sum_{m=0}^{\infty} a'_m(z-\alpha)^m$ of $f(z)$ at $z = \alpha$. Since $f(z)$ is bounded for $|z-\alpha| < 1-\alpha$, the Abel-summability of $\sum_{m=0}^{\infty} a'_m(1-\alpha)^m$ to s implies the C_1-summability of the series to the same value (see for example [2, p. 327]). The condition (iii) of Theorem 2 is also satisfied for every $\delta > 0$; hence Theorem 2 is applicable and it is sufficient to prove Theorem 2.

PROOF OF THEOREM 2. We shall use our hypotheses in the form (i), (ii′), (iii). Let $\{s'_n\}$ be the T_α-transform of $\{s_m\}$, i.e.

$$(4) \qquad s'_n = (1-\alpha)^{n+1} \sum_{m=n}^{\infty} \binom{m}{n} \alpha^{m-n} s_m = \sum_{m \geqq n} u_m(n) s_m,$$

with the notation of [5, p. 201] and $k = 1-\alpha$, so that, for each fixed value n, the maximum term $u_m(n)$ is attained for $m = [n/(1-\alpha)]$. Our idea is

[3] To obtain the matrix for the "circle method" of order α, put $k = 1-\alpha$ in [5, p. 201].

(a) to construct for each interval (m_k, M_k) in which $\{s_m\}$ is constant $=s_{m_k}$, a corresponding interval (n_k, N_k) in which $\{s'_n\}$ is almost constant and almost $=s_{m_k}$ (see equation (12));[4]

(b) to deduce from the C_1-summability of $\{s'_n\}$ the convergence of $\{s'_{n_k}\}$ and thus the convergence of $\{s_{m_k}\}$ for $k\to\infty$.

In order to attack (a) for a given $k=1, 2, \cdots$, put $\vartheta'=\vartheta/16$ and consider the intervals $I_1^{(k)}=(m_k, [m_k(1+\vartheta')]), \cdots, I_{16}^{(k)}=([m_k(1+15\vartheta')], [m_k(1+16\vartheta')])$.

For each k, let n_k be the first index n for which $[n/(1-\alpha)]$ is in $I_2^{(k)}$; such an index exists for all sufficiently large k. Similarly, let N_k be the last index n such that $[n/(1-\alpha)]$ is in $I_{15}^{(k)}$; again, such an index exists for large k. Hence we have obtained our intervals (n_k, N_k), and we note that by construction

$$\left[\frac{N_k}{1-\alpha}\right] \geqq [m_k(1+14\vartheta')]$$

and

$$[m_k(1+\vartheta')] \leqq \left[\frac{n_k}{1-\alpha}\right] \leqq [m_k(1+2\vartheta')]$$

so that

(5) $\quad N_k - n_k \geqq \kappa n_k$ for every fixed positive $\kappa < 12\vartheta'/(1+2\vartheta')$.

Consider now for any n in the interval (n_k, N_k) the T_α-transform (4) and choose a positive number $\sigma<\vartheta'/(1-\alpha)(1+\vartheta)$. For each such n the coefficients a_m in the interval $[n/(1-\alpha)]-\sigma n\leqq m\leqq [n/(1-\alpha)]+\sigma n$ vanish, for this implies $m_k<m\leqq M_k$. Therefore, putting $m=[n/(1-\alpha)]+h$, we can estimate

$$(6)\quad s'_n = \sum_{m\geqq n} u_m(n)s_m = \sum_{|h|\leqq\sigma n} u_m(n)s_m + \sum_{|h|>\sigma n} u_m(n)s_m = A_n + B_n.$$

First, let us consider A_n. We have

$$(7)\ A_n = s_{m_k}\cdot \sum_{|h|\leqq\sigma n} u_m(n) = s_{m_k}\cdot\left(1 - \sum_{|h|>\sigma n} u_m(n)\right) = s_{m_k}\cdot(1+O(e^{-\gamma n}))$$

with some $\gamma=\gamma(1-\alpha, \sigma)>0$ (notation of Theorem 139 in [5]).

Next, we estimate B_n with the use of (iii). With a constant K we obtain

$$\begin{aligned} |B_n| &\leqq K \sum_{|h|>\sigma n} u_m(n)(1+\delta)^m \\ &= K(1-\alpha)\left(\frac{1-\alpha}{\alpha}\right)^n {\sum}'\binom{m}{n}\{\alpha(1+\delta)\}^m, \end{aligned}$$

[4] Incidentally, the assumption (ii) is not needed to prove (12).

where $\sum'$ ranges over all m with $m<[n/(1-\alpha)]-\sigma n$ and $m>[n/(1-\alpha)]+\sigma n$. Put $\alpha'=(1+\delta)\alpha$ and assume that δ in (iii) is so small that

$$(1+\delta)\alpha < 1 \text{ and } 0 < \frac{1}{1-\alpha'} - \frac{1}{1-\alpha} < \frac{\sigma}{2}. \tag{8}$$

Then we have

$$|B_n| \leqq K(1-\alpha)\left(\frac{1-\alpha}{\alpha}\right)^n \sum{}''\binom{m}{n}\alpha'^m,$$

where $\sum''$ ranges over all m with $m<[n/(1-\alpha')]-(\sigma/2)n$ and $m>[n/(1-\alpha')]+(\sigma/2)n$. This sum we can estimate:

$$\sum{}''\binom{m}{n}\alpha'^m = O(e^{-\gamma n})\cdot\left(\frac{\alpha'}{1-\alpha'}\right)^n \text{ with } \gamma = \gamma\left(1-\alpha', \frac{\sigma}{2}\right),$$

so that

$$|B_n| \leqq K'\cdot\left\{\frac{1-\alpha}{1-\alpha'}\cdot\frac{\alpha'}{\alpha}\right\}^n\cdot e^{-\gamma n} \text{ with } \gamma = \gamma\left(1-\alpha', \frac{\sigma}{2}\right). \tag{9}$$

Now one can easily verify that $\gamma(1-\alpha', \sigma/2)$ can be taken as a continuous function of α' in $0<\alpha'<1$, if $\sigma>0$ is fixed. For $\alpha'\to\alpha$ it tends therefore to $\gamma(1-\alpha, \sigma/2)>0$, while the content of the braces in (9) tends to 1. Hence, if our δ was in addition to (8) small enough (depending on α and on σ and thus on α and on ϑ), we have

$$|B_n| \leqq K''\cdot e^{-\lambda_1 n} \text{ for some fixed constants } K''>0 \text{ and } \lambda_1>0; \tag{10}$$

this holds for every n in the intervals (n_k, N_k).

To bring (7) into a more suitable form, we notice that, by (iii), $s_{m_k}=O((1+\delta)^{m_k})$, and hence

$$\begin{aligned} s_{m_k}\cdot O(e^{-\gamma n}) &= O((1+\delta)^{m_k}e^{-\gamma n}) = O((1+\delta)^{n/(1-\alpha)}\cdot e^{-\gamma n}) \\ &= O(e^{-\lambda_2 n}), \end{aligned} \tag{11}$$

if δ in (iii) was given so that $\log(1+\delta)<(1-\alpha)\cdot\gamma(1-\alpha, \sigma)$. Combining (6), (7), (10), (11), we obtain

$$s_n' = s_{m_k} + O(e^{-\lambda n}) \quad \begin{array}{l}\text{for a constant } \lambda>0 \text{ and for all}\\ n \text{ in the intervals } (n_k, N_k).\end{array} \tag{12}$$

Now we come to the easier part (b) of our program. Denoting the C_1-means of $\{s_n'\}$ by σ_n', we simply write

$$
(13)\quad \begin{aligned}
\sigma'_{N_k} &= \frac{s'_0 + \cdots + s'_{N_k}}{N_k + 1} = \frac{s'_0 + \cdots + s'_{n_k}}{N_k + 1} + \frac{s'_{n_k+1} + \cdots + s'_{N_k}}{N_k + 1} \\
&= \frac{n_k + 1}{N_k + 1}\sigma'_{n_k} + \frac{N_k - n_k}{N_k + 1} s_{m_k} + O(e^{-\lambda n_k}).
\end{aligned}
$$

Assuming without loss of generality that $\sigma'_n \to 0$ $(n \to \infty)$ and remembering (5), we obtain immediately $s_{m_k} \to 0$ $(k \to \infty)$, as was to be proved.

In order to prove the extension, replace the simple device (13) by the gap theorem for C_κ-summable series (see [7, p. 469]; for a simpler proof see [6] or [3]); this gap theorem is applicable to the series $\sum(s'_n - s'_{n-1})$ because of (12), and it yields the convergence of $\{s'_{n_k}\}$ and hence the convergence of $\{s_{m_k}\}$. Thus our extension is also proved.

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