A BOUNDARY LAYER PROBLEM FOR AN ELLIPTIC EQUATION IN THE NEIGHBORHOOD OF A SINGULAR POINT

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We consider the first boundary value problem for

\[ Lu = \epsilon \Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y) \]

on a region \( R \) under the following hypotheses

I. \( R \) is an open simply- or multiply-connected region in the \((x, y)\) plane whose boundary \( \partial R \) consists of a finite number of simple closed curves, and \( R + S \) is contained in an open connected region \( R_0 \) throughout which \( A(x, y), B(x, y), C(x, y) \), and \( D(x, y) \) are of class \( C^6 \).

II. Along each closed curve of \( \partial R \) the functions giving \( x, y, \) and the boundary value \( u \) in terms of arclength are of class \( C^6 \).

III. \( C(x, y) < 0 \) on \( R_0 \).

IV. The system (for characteristics of the abridged (\( \epsilon = 0 \)) equation)

\[
\frac{dx}{dt} = -A(x, y), \quad \frac{dy}{dt} = -B(x, y)
\]  

has as its singularities on \( R + S \) a finite number of stable attractors \( P_1, \cdots, P_n \).

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We shall prove that if \( u(x, y, \epsilon) \) is the solution to our boundary value problem (existence for small \( \epsilon > 0 \) follows from results of Lichtenstein [4]), and if \( U(x, y) \) is that solution to the abridged equation
\[
L^0 U = A(x, y)U_x + B(x, y)U_y + C(x, y)U = D(x, y)
\]
which solves the initial value problem for \( R + S - P_1 - \cdots - P_n \):
\[
U = \tilde{u} \quad \text{on those portions of } S \text{ where the solutions of (1) cross into } R,
\]
then throughout \( R - P_1 - \cdots - P_n \), \( v(x, y, \epsilon) = U(x, y) - u(x, y, \epsilon) \) approaches zero as \( \epsilon \to +0 \) except possibly, as will be seen from the proof, at characteristics of (1) which are somewhere tangent to \( S \).

Now Levinson [3] has proved this in the case where (1) has no singularities on \( R \). Indeed his results show in any case that for a certain set of subregions of \( R + S \), the "regular quadrilaterals," the above stated conclusion is correct. More precisely, these "regular quadrilaterals" are defined by:

Let \( S_1 \) and \( S_2 \) be segments of curves of \( S \) having the property that they are nowhere tangent to a characteristic of (2) and being so related that those characteristics of (2) emanating from \( S_1 \) pass out of \( R \) on \( S_2 \) and conversely. Here \( S_1 \) signifies that one of the pair of segments across which these characteristics cross into \( R \) (referring to (1)). That closed simply-connected subregion of \( R + S \) bounded by \( S_1, S_2 \), and the two characteristics of (2) joining their endpoints is a "regular quadrilateral." [Thus in our problem \( R \) cannot be decomposed as a union of regular quadrilaterals.]

Levinson's result then reads (in our notation):

**Theorem.** In a regular quadrilateral we may write
\[
v(x, y, \epsilon) = z(x, y, \epsilon) - w(x, y, \epsilon)
\]
where
\[
w = O(\epsilon^{1/2}) \quad \text{as } \epsilon \to +0
\]
uniformly in the quadrilateral and \( w = 0 \) on \( S_1 \) and \( S_2 \), and where \( z(x, y, \epsilon) \) has near and on \( S_2 \) the form \( e^{-g(x, y)} h(x, y) \). Here \( g = 0 \) on \( S_2 \) and \( g > 0 \) off of \( S_2 \) and both \( g \) and \( h \) are of class \( C^2 \); moreover, at points of the quadrilateral where the above representation is not valid
\[
z = O(\epsilon^{-\delta/\epsilon}) \quad \text{uniformly as } \epsilon \to +0
\]
for a fixed positive \( \delta \).

Therefore it will suffice for us to prove that the stated result holds for a second set of subregions of \( R + S \), the "regular triangles," these being defined as follows:
Let \( S_0 \) be a closed segment of a curve of \( S \) having the property that it is nowhere tangent to a characteristic of (2) and such that those characteristics emanating from \( S_0 \) enter into and remain in \( R \) (referring to (1)), where they approach one of the \( P_1 \). That simply-connected subregion of \( R+S \) traced out by the characteristics emanating from \( S_0 \) is a "regular triangle."

Moreover to show that, on any regular triangle \( T \), \( v(x, y, \varepsilon) \) approaches zero as \( \varepsilon \to +0 \) it will clearly be sufficient to show that \( v(x, y, \varepsilon) \) approaches zero as \( \varepsilon \to +0 \) on all subregions \( G \) of the following type:

\( G \) is a simply-connected subregion of \( T \) which is bounded by \( S_0 \), by an orthogonal trajectory to those characteristics of (2) lying in \( T \) which intersects the two characteristics making up the "sides" of \( T \) but does not intersect \( S_0 \), and by the requisite portions of the "sides" of \( T \).

We note, too, that the nontangency condition and the stability of the attractor allow us to consider two other triangles \( T_1, T_2 \) such that \( T_2 \supset T_1 \supset T \) and corresponding subregions \( G_1, G_2 \) defined analogously to \( G \) (the orthogonal trajectory boundary for \( G_1 \) is taken to be a portion of that for \( G_2 \) and it lies "nearer" to the attractor than does that for \( G \), so that \( G_2 \supset G_1 \supset G \)). We shall find it convenient to introduce characteristic coordinates on \( G_2 \). Levinson [3] has shown that there are \( C^5 \) functions \( \sigma(x, y), \tau(x, y) \) satisfying \( A\sigma_x + B\sigma_y = 0, B\tau_x - A\tau_y = 0 \) on a region such as \( G_2 \), such that: \( \partial(\sigma, \tau)/\partial(x, y) \neq 0 \); along characteristics of (2), \( \sigma \) = constant; and the curvilinear coordinates \( (\sigma, \tau) \) are orthogonal. In addition \( A\tau_x + B\tau_y < 0 \) if \( \tau \) is taken as increasing toward the singularity, as we shall do. We denote the values of \( \sigma \) on the characteristic boundaries of \( G_2 \) by \( \sigma_1 \) and \( \sigma_2 \) while those values on the characteristic boundaries of \( G_1 \) are denoted by \( \sigma_1 \) and \( \sigma_2 \) (ordering so that \( \sigma_2 > \sigma_1 > \sigma_1 > \sigma_1 \)).

The proof proceeds in the following manner: Following a technique used by Kamenomostskaya [2] and Aronson [1], we exhibit functions \( W_1 \) and \( W_2 \) defined on \( G_1 \) which consist of "boundary layer" terms alone, except for terms which are \( O(\varepsilon^{1/2}) \), and which satisfy

\[
LW_1 = Lv \quad LW_2 = -Lv = L(-v),
\]

where we recall

\[
v = U - u.
\]

These functions are so chosen that \( W_1, W_2 > |v| \) on the boundary of \( G_1 \) for sufficiently small \( \varepsilon > 0 \). Use of the maximum principle then implies that for such \( \varepsilon \), \( W_1 > v \) and \( W_2 > -v \) throughout \( G_1 \). Finally, in-
spection of $W_1$ and $W_2$ shows that they are uniformly $O(\varepsilon^{1/2})$ on $G$, and this yields the desired result.

Now in the preceding outline of the proof, "boundary layer" terms denote functions $H(x, y, \varepsilon)$ of the following type:

1. In a neighborhood of a portion of the boundary of the region, $H(x, y, \varepsilon)$ is of the form $e^{-\theta(x,y)/\varepsilon^m}h(x,y)$, where $m$ is a positive constant and where $g, h$ are of class $C^2$, $g$ being positive except on this portion of the boundary. Moreover $H$ is of class $C^2$ throughout the entire region and $H$ is uniformly $o(1)$ as $\varepsilon \to +0$, except in the boundary neighborhood.

2. $LH = o(1)$ as $\varepsilon \to +0$, uniformly in the entire region.

From direct substitution it is readily seen that only for $m = 1/2$ and $m = 1$ does the second condition give rise to as few as two equations which the functions $g$ and $h$ must satisfy. For these values the equations to be satisfied are:

For $m = 1$:

$$g_x + g_y - Ag_x -Bg_y = 0,$$

$$(A - 2g_x)h + (B - 2g_y)h + (C - \Delta g)h = 0,$$

or, equivalently,

$$g_x^2 + g_y^2 - Ag_x - Bg_y = 0,$$

and that except for the boundary neighborhood involved $H$ is uniformly $O(\varepsilon^{-\delta/\varepsilon})$ where $\delta$ is a fixed positive constant. For the case $m = 1/2$, on the other hand, it follows from (3) that $g$ must simply be a function which is constant on each characteristic of (2), so that to obtain a "boundary layer" term by this scheme we require a characteristic boundary (that is, we can only require $g = 0$ on a boundary which is characteristic if we are to retain a bona fide boundary layer form). Moreover since $h$ satisfies a linear equation (cf. (3)) whose characteristics coincide with those of (2) we may readily specify this function throughout $G_1$ as a solution to an initial value problem. It
follows that in this case we can obtain a boundary layer term having the indicated exponential form throughout the region $G_1$.

We now consider a pair of functions $W_1$ and $W_2$ defined on $G_1$ and having the form:

\begin{equation}
W_i(\sigma, \tau, \epsilon) = H^{(0)}(\sigma, \tau) + h^{(1)}(\sigma, \tau) \left[ e^{-k(\sigma - \sigma_i)^{1/2}} + e^{-k(\sigma - \epsilon)^{1/2}} \right] + \epsilon^{1/2} Z_i,
\end{equation}

$i = 1, 2$.

In the above expression $H^{(0)}(\sigma, \tau, \epsilon)$ is chosen according to Levinson's method for the case $m=1$ to be a boundary layer term for the region $G_2$ corresponding to the following values on the orthogonal trajectory boundary (a curve $\tau = \text{constant}$) of $G_2$

\begin{align*}
M + 1, & \quad \sigma_1 < \sigma < \sigma_2, \\
0, & \quad \sigma_1 < \frac{\sigma_1 + \sigma_2}{2}, \quad \sigma > \frac{\sigma_1 + \sigma_2}{2},
\end{align*}

\begin{equation}
H^{(0)} = \begin{cases} 
M + 1, & \quad \sigma_1 < \sigma < \sigma_2, \\
0, & \quad \sigma_1 < \frac{\sigma_1 + \sigma_2}{2}, \quad \sigma > \frac{\sigma_1 + \sigma_2}{2}, \\
a \text{positive } C^2 \ “\text{tieup}\” \text{ function} & \quad \frac{\sigma_1 + \sigma_2}{2} \leq \sigma \leq \sigma_1, \\
& \quad \frac{\sigma_2 + \sigma_2}{2} \leq \sigma \leq \sigma_2
\end{cases}
\end{equation}

$[M$ is a uniform bound for $v(x, y, \epsilon)$ on $G_2$—use of the maximum principle extended to the inhomogeneous case $[3]$ shows there is a uniform bound for $u(x, y, \epsilon)]$. As for $h^{(1)}(\sigma, \tau)$, we choose it to be that solution of (cf. (3))

\begin{equation}
(A \tau_x + B \tau_y) h_T + \left[ C + k^2(\sigma_x^2 + \sigma_y^2) \right] h = 0
\end{equation}

which on the segment of $S$ bounding $G_1$ takes on the value $M+1$, while $k$ is a positive constant required to be sufficiently large so that the coefficient of $h$ in (6) is positive throughout $G_1$. Thus $h^{(1)}(\sigma, \tau)$ is defined and $h^{(1)} \geq M+1$ throughout $G_1$. Finally, we choose the $Z_i$ so that

\begin{align*}
LW_1 &= L v = \epsilon \Delta U, \quad LW_2 = L(-v) = -\epsilon \Delta U,
\end{align*}

and $Z_1$ and $Z_2$ vanish on the entire boundary of $G_1$. Thus the $Z_i$ must solve the homogeneous boundary value problem on $G_1$ for

\begin{align*}
\epsilon^{1/2} L_{Z_i} &= \epsilon^{1/2} \left\{ 2k h_x^{(1)} (\sigma_x + \sigma_y) + k h^{(1)} \Delta \sigma \right. \\
& \quad \times \left( e^{-k(\sigma - \sigma_i)^{1/2}} - e^{-k(\sigma - \epsilon)^{1/2}} \right) \\
& \quad + \epsilon \left[ h_x^{(1)} (\sigma_x + \sigma_y) + h^{(1)} (\tau_x + \tau_y) \\
& \quad + h^{(1)} \Delta \sigma + h^{(1)} \Delta \tau ) (e^{-k(\sigma - \sigma_i)^{1/2}} - e^{-k(\sigma - \epsilon)^{1/2}}) \right] \\
& \quad - LH^{(0)} + \epsilon (-1)^{i+1} \Delta U, \quad i = 1, 2.
\end{align*}
(It follows from results of Lichtenstein that $C^2$ solutions to these boundary value problems exist [4].) It therefore follows from the inhomogeneous case maximum principle [3] that $Z_i = O(1)$ as $\epsilon \to +0$, uniformly on $G_1$ [recalling that $LH^{(0)} = O(\epsilon)$].

Now examination of the functions $W_1$ and $W_2$ indicates that in $G$

$W_i = O(\epsilon^{1/2})$ uniformly as $\epsilon \to +0$. Moreover $W_1 - v$ and $W_2 + v$ satisfy the homogeneous equation $Lu = 0$, while investigation of the boundary values shows that, for sufficiently small $\epsilon$, $W_1 - v$ and $W_2 + v$ are positive throughout the boundary of $G_1$. Thus the maximum principle shows that for these $\epsilon$, $W_1 - v \geq 0$ and $W_2 + v \geq 0$ throughout $G_1$. In particular,

$$|v| \leq W_1 + W_2 = O(\epsilon^{1/2})$$

throughout $G$,

which completes the proof.

**Bibliography**


3. N. Levinson, *The first boundary value problem for $\epsilon\Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$ for small $\epsilon$*, Ann. of Math. vol. 51 (1950) pp. 428–445.


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