ON H-SPACES WITH TWO NONTRIVIAL HOMOTOPY GROUPS[1]

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Introduction. Although several topologists (e.g. H. Hopf and A. Borel) have found necessary algebraic conditions for a space to admit an H-space structure, very little has been done towards obtaining sufficient conditions. The author believes that the present paper contains essentially the first result in the latter direction.

Let Y be a topological space with $y_0 \in Y$, $Y \vee Y = Y \times y_0 \cup y_0 \times Y \subset Y \times Y$. If $\phi: Y \vee Y \to Y$ is the map given by $\phi(y, y_0) = (y_0, y) = y$, then the problem of finding an H-space structure on Y may be expressed as the problem of extending ϕ to a map $\phi': Y \times Y \to Y$. It is found that if Y is a 1-connected, locally finite CW-complex [**3**], the obstructions to extending ϕ may be expressed in terms of Postnikov invariants [**4**] and partial extensions of ϕ. If Y has only two nonzero homotopy groups then there is at most one nontrivial obstruction. This will be zero if and only if the Eilenberg-MacLane k-invariant of Y is primitive.

The relation between the existence of an H-structure and the vanishing of the J. H. C. Whitehead bracket products is investigated. This leads to a description of the lowest-dimensional bracket products on spaces whose first two nontrivial homotopy groups are in dimensions n and $2n-1$ $(n>1)$.

1. This section presents much of the notation to be used, some discussion preliminary to the main result of the paper and an example of a space which is not an H-space, and yet has trivial bracket products.

If $f_i: X_i \to Y_i$ $(i=1, 2)$ are maps then the map $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for $x_i \in X_i$. If $g_1: Y_1 \to Z_1$, then $g_1 \circ f_1: X_1 \to Z_1$ is the map given by $g_1 \circ f_1(x) = g_1(f_1(x))$ for $x \in X$, $I = [0, 1]$ is the closed unit interval; $I^m = I \times \cdots \times I$ (m-factors) is the unit m-cube; $\dot{I}^m$ is the usual boundary set of I^m. The discussion is restricted to locally finite CW-complexes, for if Y is a locally finite CW-complex then $Y \times Y$ is a CW-complex whose (closed) m-cells may be taken to be of the form $E^m = E^p \times E^q$ $(p+q=m$; E^p, E^q are, respectively, p-, and q-cells of $Y)$ [**6**]. The characteristic map of E^m is $e^m = e^p \times e^q$; $I^m \to E^m$ where e^p, e^q are character-

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istic maps of E^p, E^q and I^m is identified with $I^p \times I^q$. Given cochains $c^p \in C^p(Y; G_1)$, $c^q \in C^q(Y; G_2)$ and a pairing $\xi: G_1 \otimes G_2 \to G$ we form a cochain $c^p \times c^q \in C^{p+q}(Y \times Y; G)$ whose value on a cell $E^p \times E^q$ is $c^p \times c^q(E^p \times E^q) = \xi(c^p(E^p) \otimes c^q(E^q))$ and which is zero elsewhere. If c^p, c^q are cocycles then $c^p \times c^q$ is, and the corresponding cohomology classes are $[c^p]$, $[c^q]$, $[c^p] \times [c^q] = [c^p \times c^q]$. If X is a CW-complex, its m-skeleton will be designated by X^m. The map $\phi: Y \vee Y \to Y$, presented in the introduction, will be called the *folding map* of Y. If $f: A \to B$ is a map, its homotopy class will be designated by $\{f\}$.

If the folding map has been extended over $Y \vee Y \cup (Y \times Y)^m$ then the obstruction to extending over the $(m+1)$-skeleton is in the cohomology group $H^{m+1}(Y \times Y, Y \vee Y; \pi_m(Y))$.

In case Y is $(n-1)$-connected, $(Y \times Y, Y \vee Y)$ will be $(2n-1)$-connected.

PROPOSITION 1. *The obstruction in dimension* $2n$ *to extending the folding map is*

$$d^n \times d^n \in H^{2n}(Y \times Y, Y \vee Y; \pi_{2n-1}(Y))$$

where $d^n \in H^n(Y, y; \pi_n(Y))$ *is the characteristic class for* Y *and the pairing of* $\pi_n(Y) \otimes \pi_n(Y)$ *into* $\pi_{2n-1}(Y)$ *is the J. H. C. Whitehead bracket product.*

PROOF. We may replace Y by a space whose $(n-1)$-skeleton is a point (see §3). Then the $(2n-1)$-skeleton of $Y \times Y$ is contained in $Y \vee Y$ so that the obstruction cocycle to extending ϕ is given by:

$$\overset{2n}{c}(\overset{n}{E_1} \times \overset{n}{E_2}) = \{\phi \circ (\overset{n}{e_1} \times \overset{n}{e_2}) \mid \dot{I}^{2n}\}$$

where E_1^n, E_2^n are cells of Y and e_1^n, e_2^n their characteristic maps. Note that

$$\{\phi \circ (\overset{n}{e_1} \times \overset{n}{e_2}) \mid \dot{I}^{2n}\} = [\{\overset{n}{e_1}\}, \{\overset{n}{e_2}\}]$$

with $\{e_1^n\}$, $\{e_2^n\}$ regarded as elements of $\pi_n(Y)$ [1]. But d_n is the class of the cocycle c^n given by

$$c^n(E^n) = \{e^n\}.$$

Thus $c^{2n} = c^n \times c^n$.

COROLLARY 2. $c^{2n} = 0$ *if and only if* $[\alpha, \beta] = 0$ *for* $\alpha, \beta \in \pi_n(Y)$.

PROOF. Suppose $c^{2n} = 0$. Since $H_n(Y) \approx \pi_n(Y)$ the characteristic maps of n-cells generate $\pi_n(Y)$. Thus $[\alpha, \beta] = 0$ when α, β are in this set of generators; hence $[\alpha', \beta'] = 0$ for all $\alpha', \beta' \in \pi_n(Y)$. The converse is trivial.

On the other hand, if Y is an H-space then all bracket products vanish. This raises the question: if Y is a CW-complex and $[\pi_p(Y), \pi_q(Y)]=0$ for $p, q>0$, is Y an H-space? The answer is negative, and we present a counter-example:

We construct a CW-complex, K, by specifying its m-skeleta, K^m. Let $n>2$ be an integer and p an odd prime. The $2n$-skeleton of K is taken to be the $2n$-skeleton of a CW-complex which is an Eilenberg-MacLane space of type (Z_p, n), where Z_p is the integers mod p. There are no cells in dimension $2n+1$ $(K^{2n+1}=K^{2n})$ and in higher dimensions, cells are appended so that $\pi_i(K)=0$ for $i>2n$ [7].

This creates a space whose homotopy groups are trivial except in dimensions n and $2n$. Thus all bracket products vanish. Also note that $H^n(K; Z_p)\approx H^{n+1}(K; Z_p)\approx Z_p$ since the cohomology groups in these dimensions are those of a space of type (Z_p, n). The $(2n+1)$-cohomology group is zero, since there are no $(2n+1)$-cells.

Now, K is not an H-space, for if it were, its cohomology ring would be a Hopf algebra and the cup product of an element of $H^n(K; Z_p)$ with an element of $H^{n+1}(K; Z_p)$ would be nonzero, contradicting $H^{2n+1}(K; Z_p)=0$ [2].

2. **The main result.** Let Y be a 1-connected CW-complex, and suppose that the folding map has been extended to $\phi: Y\vee Y\cup(Y\times Y)^m \to Y$. Let X be a CW-complex consisting of Y united with i-cells, E^i, $(i>m)$ such that $\pi_i(X)=0$ for $i\geqq m$. Note that below dimension m, the inclusion map induces isomorphisms of the homotopy groups of X and Y.

PROPOSITION 3. *X is an H-space.*

PROOF. The m-skeleta of X and Y are the same, so that $(X\times X)^m = (Y\times Y)^m$. Thus ϕ provides an extension $\psi': X\vee X\cup(X\times X)^m\to X$. But $H^{i+1}(X\times X, X\vee X; \pi_i(X))=0$ for $i>m$, whence all obstructions to extending ψ' vanish. One such extension, let us call it $\psi: X\times X\to X$, is chosen for the structure map of X.

The condition $\pi_i(X)=0$ for $i\geqq m$ also implies that any two extensions of ψ' will be homotopic.

In the diagram below, $i_1^*, \cdots, i_5^*$ are homomorphisms induced by the appropriate inclusion maps: ψ^*, ψ_1^* are induced by ψ. The coefficient group for each of these cohomology groups is $\pi_m(Y)$. This symbol has been omitted to save space. It is well known that i_1^* sends $H^{m+1}(X\times X, X\vee X; \pi_m(Y))$ isomorphically onto a direct summand of $H^{m+1}(X\times X; \pi_m(Y))$. This direct sum decomposition induces the homomorphism ρ. The composition $\rho\circ i_1^*$ is the identity automorphism of $H^{m+1}(X\times X, X\vee X; \pi_m(Y))$, and the kernel of ρ is essentially $H^{m+1}(X \vee X;\, \pi_m(Y))$. The diagram is commutative.

$$\begin{array}{ccccc}
 & \overset{i_1^*}{\swarrow}\!\!\overset{\rho}{\nearrow} & H^{m+1}(X+X,\, X\vee X) & \overset{i_2^*}{\searrow} & \\
H^{m+1}(X\times X) & \underset{i_3^*}{\leftarrow} & H^{m+1}(X\times X,\, Y\vee Y) & \underset{i_4^*}{\rightarrow} & H^{m+1}(Y\times Y,\, Y\vee Y) \\
\psi_* \uparrow & & \uparrow \psi_1^* & & \\
H^{m+1}(X) & \underset{i_5^*}{\leftarrow} & H^{m+1}(X,\, Y) & &
\end{array}$$

Let $k' \in H^{m+1}(X,\, Y;\, \pi_m(Y))$ be the first obstruction to retracting X onto Y, and $k = i_5^* k' \in H^{m+1}(X;\, \pi_m(Y))$. The maps $p_1, p_2 \colon X\times X \to X$ are the projections $p_i(x_1, x_2) = x_i$ for $x_i \in X$, $i = 1, 2$.

PROPOSITION 4. *Let $\gamma \in H^{m+1}(Y\times Y,\, Y\vee Y;\, \pi_m(Y))$ be the class of the obstruction to extending ϕ to $(Y\times Y)^{m+1} \cup Y \vee Y$. Then*

$$i_2^* \rho(\psi^* - p_1^* - p_2^*)k = \gamma.$$

PROOF. The cohomology class $\psi_1^* k'$ is the first obstruction to extending $(\psi \mid Y\vee Y \cup (X\times X)^m) = (\phi \mid Y \vee Y \cup (Y\times Y)^m)$ to $X\times X$ and hence $i_4^*\psi_1^* k' = \gamma$. On the other hand, $i_3^*\psi_1^* k' = \psi^* i_5^* k' = \psi^* k$, and so $i_2^*\rho\psi^* k = i_2^*\rho i_3^* \psi_1^* k' = i_4^*\psi_1^* k' = \gamma$. Finally, $\rho \circ (p_1^* + p_2^*)$ is the trivial homomorphism, whence $i_2^*\rho(\psi^* - p_1^* - p_2^*)k = i_2^*\rho\psi^* k = \gamma$.

If W is an H-space with $\psi, p_1, p_2 \colon W\times W \to W$ the structure map and the two projections respectively, then a cohomology class, u, is called *primitive* whenever $(\psi^* - p_1^* - p_2^*)u = 0$.

THEOREM 5. *The obstruction class, γ, vanishes if and only if k is primitive.*

PROOF. We already have $i_2^*\rho(\psi^* - p_1^* - p_2^*)k = \gamma$ so that if k is primitive, $\gamma = 0$. To obtain the converse, we first note that i_2^* is an isomorphism since the inclusion map of Y in X induces isomorphisms $H^i(X) \approx H^i(Y)$ for $i \leqq m$. But the image of $(\psi^* - p_1^* - p_2^*)k$ in $H^{m+1} \cdot (X \vee X;\, \pi_m(Y))$ is zero, since $(\psi \mid X \vee X)^* = (p_1 \mid X\vee X)^* + (p_2 \mid X \vee X)^*$. Thus $(\psi^* - p_1^* - p_2^*)k \in i_1^* H^{m+1}(X\times X,\, X \vee X;\, \pi_m(Y))$ from which it follows that $\rho(\psi^* - p_1^* - p_2^*)k = 0$ implies $(\psi^* - p_1^* - p_2^*)k = 0$. This completes the proof.

Note that k is essentially the $(m+1)$-Postnikov invariant of Y. The theorem fails to provide a decisive victory over the problem of characterizing H-spaces which are CW-complexes, inasmuch as it depends upon choosing a particular extension, ψ, in each dimension. However, for sufficiently simple spaces the problem can be solved:

THEOREM 6. *Suppose Y is a CW-complex which has only two nontrivial homotopy groups, $\pi_n(Y)$ and $\pi_m(Y)$, with $1 < n < m$. Then Y is* *an H-space if and only if the Eilenberg-MacLane k-invariant of Y is primitive.*

PROOF. The only nontrivial obstruction to extending the folding map is in $H^{m+1}(Y\times Y,\ Y\vee Y;\ \pi_m(Y))$. Thus the space X is of type $(\pi_n(Y), n)$ and k may be identified with the k-invariant, $k_n^{m+1}(Y)$. The result now follows from Theorem 4.

It may now be seen that the example of §1 was obtained by constructing a space with nonprimitive k-invariant.

In §3 it is shown that given abelian groups π_n, π_m and an element $k\in H^{m+1}(\pi_n, n; \pi_m)$, there is a space Y (which may be taken to be a CW-complex) such that $\pi_i(Y)=0$ for $i\neq n, m$, $\pi_n(Y)=\pi_n$, $\pi_m(Y)=\pi_m$ and $k_n^{m+1}(Y)=k$. Any two such CW-complexes are of the same homotopy type. This observation and Theorem 5 give a classification of CW-complexes which admit H-structures and have only two nonvanishing homotopy groups.

Theorem 5 and Proposition 1 (§1) may be combined to yield a result about the Whitehead bracket product. Suppose $\pi_i(Y)=0$ for $0\leqq i<n$ and $n<i<2n-1$. Let $\pi_n=\pi_n(Y)$ and $\pi_{2n-1}=\pi_{2n-1}(Y)$; $\pi_n\otimes\pi_n$, $\pi_n\oplus\pi_n$ designate respectively the tensor product and the direct sum of π_n with itself. Three homomorphisms, $\bar\psi$, $\bar p_1$, $\bar p_2:\pi_n\oplus\pi_n\to\pi_n$ are defined by $\bar\psi(\alpha, \beta)=\alpha+\beta$, $\bar p_1(\alpha, \beta)=\alpha$, $\bar p_2(\alpha, \beta)=\beta$ for α, $\beta\in\pi_n$.

PROPOSITION 7. *The cohomology class* $(\bar\psi^*-\bar p_1^*-\bar p_2^*)k_n^{2n}(Y)\in H^{2n}(\pi_n\oplus\pi_n, n; \pi_{2n-1})$ *defines a homomorphism* $W:\pi_n\otimes\pi_n\to\pi_{2n-1}$ *such that* $W(\alpha\otimes\beta)=[\alpha, \beta]$ *for* α, $\beta\in\pi_n$.

PROOF. Consider the diagram,

$$\begin{array}{ccc}
\operatorname{Hom}\{\pi_n\otimes\pi_n;\ \pi_{2n-1}\} & \xleftarrow{\theta_1} & H^{2n}(\pi_n\oplus\pi_n,\ n;\ \pi_{2n-1}) \\
\Big\uparrow & & \Big\downarrow \kappa^* \\
(\eta\otimes\eta)^* & & H^{2n}(X\times X;\ \pi_{2n-1}) \\
 & & \Big\uparrow i_1^*\circ(i_2^*)^{-1} \\
\operatorname{Hom}\{H_n(Y)\otimes H_n(Y);\ \pi_{2n-1}\} & \xleftarrow[\theta_2]{} & H^{2n}(Y\times Y,\ Y\vee Y;\ \pi_{2n-1})
\end{array}$$

The space X is of type (π_n, n), so that the natural chain maps from the cell complex of X into the singular complex of X and thence into $K(\pi_n, n)$ induces a chain map, κ, from the cell complex of $X\times X$ into $K(\pi_n\oplus\pi_n, n)$. This last induces the isomorphism κ^*. The homomorphisms $\bar\psi$, $\bar p_1$, $\bar p_2$ are algebraic analogues of ψ, p_1, p_2. In particular, $\kappa^*(\bar\psi^*-\bar p_1^*-\bar p_2^*)k_n^{2n}(Y)=(\psi^*-p_1^*-p_2^*)k$. The homomorphisms i_1^*, i_2^*

are as defined in Theorem 5; $\eta:\pi_n\to H_n(Y)$ is the Hurewicz isomorphism and $(\eta\otimes\eta)^*$ is the induced isomorphism of the Hom groups. The Kunneth formula yields the homomorphisms θ_1, θ_2; θ_2 is an isomorphism. Note that $\theta_1\circ(\kappa^*)^{-1}\circ i_1^*\circ(i_2^*)^{-1}\circ\theta_2^{-1}\circ((\eta\otimes\eta)^*)^{-1}$ is the identity automorphism of Hom $\{\pi_n\otimes\pi_n;\ \pi_{2n-1}\}$ onto itself.

Recall that $d_n\in H^n(Y;\ \pi_n)$ is the basic cohomology class of Y. Thus the image of d_n in Hom $\{H_n(Y);\ \pi_n\}$ is η^{-1} and $\theta_1(W\circ(\eta^{-1}\otimes\eta^{-1}))=d_n\times d_n=\gamma$. We now have,

$$\begin{aligned} W &= \theta_1\circ(\kappa^*)^{-1}\circ \overset{*}{i_1}\circ(\overset{*}{i_2})^{-1}\circ\theta_2^{-1}\circ((\eta\otimes\eta)^*)^{-1}(W)\\ &= \theta_1\circ(\kappa^*)^{-1}\circ \overset{*}{i_1}\circ(\overset{*}{i_2})^{-1}(\gamma)\\ &= \theta_1\circ(\bar{\psi}^*-\overset{*}{\bar{p}_1}-\overset{*}{\bar{p}_2})\,k_n^{2n}(Y). \end{aligned}$$

PROPOSITION 8. *Suppose Y is a CW-complex with only two nonvanishing homotopy groups, $\pi_n=\pi_n(Y)$ and $\pi_m=\pi_m(Y)$. Then there is a space of loops Ω having the same homotopy groups and k-invariant as Y if and only if $k_n^{m+1}(Y)$ is the suspension of an element*

$$k^{m+2}\in H^{m+2}(\pi_n,\ n+1;\ \pi_m).$$

PROOF. Let S be the suspension homomorphism and suppose $k_n^{m+1}(Y)=Sk^{m+2}$. Let W be a space such that $\pi_i(W)=0$ $(i\neq n+1,\ m+1)$, $\pi_{n+1}(W)=\pi_n$, $\pi_{m+1}(W)=\pi_m$, $k_{n+1}^{m+2}(W)=k^{m+2}$. Then $\Omega=$ the space of loops on W is the desired space. The converse is immediate.

J. C. Moore has demonstrated (unpublished) that if $\alpha\in H^{m+1}(\pi,\ n;\ G)$ is primitive then α is the suspension of an element of $H^{m+2}(\pi,\ n+1;\ G)$. Thus all H-spaces of the type under discussion are essentially spaces of loops.

3. Let X be a given 0-connected space, $x_0\in X$. We construct a CW-complex, $K(X)$, whose 0-skeleton is a point $k_0\in K(X)$, and a map $f(X):K(X)\to X$ such that $f(X)$ induces isomorphisms $f(X)_\#$: $\pi_i(K(X),\ k_0)\to\pi_i(X,\ x_0)$ for $i\geqq 0$. This is done by specifying the n-skeleta K^n of $K(X)$ and maps $f_n:K^n\to X$.

The 0-skeleton consists of one cell $E^0=k_0$. Suppose K^n and $f_n:K^n\to X$ are constructed such that the induced homomorphism

$$f_{n\#}^{(i)}:\pi_i(K^n,\ k_0)\to\pi_i(X,\ x_0)$$

is an isomorphism for $i<n$ and onto for $i=n$. Let A_{n+1} be the kernel of $f_{n\#}^{(n)}$ and $B_{n+1}\subset\pi_{n+1}(X,\ x_0)$ a set of generators of $\pi_{n+1}(X,\ x_0)$. Append cells $E_\alpha^{n+1}(\alpha\in A_{n+1})$, so that if e_α^{n+1} is the characteristic map of E_α^{n+1} then $(e_\alpha^{n+1}|\dot{I}^{n+1})\in\alpha$, and cells E_β^{n+1} $(\beta\in B_{n+1})$ with e_β^{n+1} $(\dot{I}^{n+1})=k_0$. The map f_n can be extended over cells E_α^{n+1} $(\alpha\in A_{n+1})$ since

$f_n | e_\alpha^{n+1}$ $(\dot{I}^{n+1})$ is null-homotopic; $f_{n+1} | E_\beta^{n+1}$ $(\beta \in B_{n+1})$ is determined by $f_{n+1} \circ e_\beta^{n+1} \in \beta$. Then

$$f_{n+1\#}^{(i)} : \pi_i(K^{n+1}, k_0) \to \pi_i(X, x_0)$$

is an isomorphism for $i < n+1$ and onto for $i = n+1$.

The complex $K(X)$ is $\bigcup_{n=0}^\infty K^n$, with the topology: C is closed in $K(X)$ if and only if $C \cap K^n$ is closed in K^n for each n. The map $f(X) : K(X) \to X$ is given by $(f(X) | K^n) = f_n$. Note that this is a modification of a construction by J. H. C. Whitehead [7].

Proposition 9. *If π_n, π_m are abelian groups $(n < m)$ and $k \in H^{m+1}(\pi_n, n; \pi_m)$ then there is a CW-complex K such that $\pi_i(K) = 0$ for $i \neq n, m$, $\pi_n(K) = \pi_n$, $\pi_m(K) = \pi_m$, $k^m = k$.*

Proof. Let E be the space of paths in the Eilenberg-MacLane space $K(\pi_m, m+1)$ terminating in some point $y_0 \in K(\pi_m, m+1)$ with fibre map $p_1 : E \to K(\pi_m, m+1)$ and fibre $K(\pi_m, m)$. If $d \in H^{m+1}(\pi_m, m+1; \pi_m)$ is the basic cohomology class, then there is a map $f : K(\pi_n, n) \to K(\pi_m, m+1)$ such that $f^*(d) = k \in H^{m+1}(\pi_n, n; \pi_m)$. Note that $K(\pi_m, m+1)$, $K(\pi_n, n)$ may be chosen to be CW-complexes and f cellular. The map f induces a space X and maps p_2, F such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & E \\ p_2 \downarrow & & \downarrow p_1 \\ K(\pi_n, n) & \rightarrow & K(\pi_m, m+1) \end{array}$$

is commutative and X is a fibre space over $K(\pi_n, n)$ with fibre map p_2 and fibre $K(\pi_m, m)$. From the homotopy sequence of the fibre map p_2 we see that $\pi_i(X) = 0$ for $i \neq n, m$, $\pi_n(X) = \pi_n$, $\pi_m(X) = \pi_m$.

We know that there is a map, j, of the m-skeleton of $K(\pi_n, n)$ into X such that $p_2 \circ j$ is the identity, and that the obstruction to extending j is k_n^{m+1}. If E^{m+1} is an $(m+1)$-cell of $K(\pi_n, n)$ then its characteristic map e^{m+1} (considered as a null-homotopy of $(e^{m+1} | \dot{I}^{m+1})$) can be lifted to a map $g : \dot{I}^{m+1} \times I \to X$ with $(g | \dot{I}^{m+1} \times 0) = j \circ (e^{m+1} | \dot{I}^{m+1})$ and $g' = (g | \dot{I}^{m+1} \times 1) : \dot{I}^{m+1} \to K(\pi_m, m)$. If ∂ is the boundary homomorphism of the homotopy sequence of p_1 and $F' = (F | K(\pi_m, m))$ then

$$\partial^{-1} F^1_{\#} \{g'\} = \{f \circ e^{m+1}\} \in \pi_{m+1}(K(\pi_m, m+1)).$$

But $f^* d(E^{m+1}) = \{f \circ e^{m+1}\}$. Thus there is an isomorphism of $H^{m+1}(\pi_n, n; \pi_m(X))$ onto $H^{m+1}(\pi_n, n; \pi_{m+1}(K(\pi_m, m+1)))$ carrying

k_n^{m+1} into $k=f^*d$. The desired CW-complex is then obtained as in the beginning of this section.

This proof is the obvious generalization of one given by Thom [5].

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