

# ON $H$ -SPACES WITH TWO NONTRIVIAL HOMOTOPY GROUPS<sup>1</sup>

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**Introduction.** Although several topologists (e.g. H. Hopf and A. Borel) have found necessary algebraic conditions for a space to admit an  $H$ -space structure, very little has been done towards obtaining sufficient conditions. The author believes that the present paper contains essentially the first result in the latter direction.

Let  $Y$  be a topological space with  $y_0 \in Y$ ,  $Y \vee Y = Y \times y_0 \cup y_0 \times Y \subset Y \times Y$ . If  $\phi: Y \vee Y \rightarrow Y$  is the map given by  $\phi(y, y_0) = (y_0, y) = y$ , then the problem of finding an  $H$ -space structure on  $Y$  may be expressed as the problem of extending  $\phi$  to a map  $\phi': Y \times Y \rightarrow Y$ . It is found that if  $Y$  is a 1-connected, locally finite CW-complex [3], the obstructions to extending  $\phi$  may be expressed in terms of Postnikov invariants [4] and partial extensions of  $\phi$ . If  $Y$  has only two nonzero homotopy groups then there is at most one nontrivial obstruction. This will be zero if and only if the Eilenberg-MacLane  $k$ -invariant of  $Y$  is primitive.

The relation between the existence of an  $H$ -structure and the vanishing of the J. H. C. Whitehead bracket products is investigated. This leads to a description of the lowest-dimensional bracket products on spaces whose first two nontrivial homotopy groups are in dimensions  $n$  and  $2n - 1$  ( $n > 1$ ).

1. This section presents much of the notation to be used, some discussion preliminary to the main result of the paper and an example of a space which is not an  $H$ -space, and yet has trivial bracket products.

If  $f_i: X_i \rightarrow Y_i$  ( $i=1, 2$ ) are maps then the map  $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$  is defined by  $f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2))$  for  $x_i \in X_i$ . If  $g_1: Y_1 \rightarrow Z_1$ , then  $g_1 \circ f_1: X_1 \rightarrow Z_1$  is the map given by  $g_1 \circ f_1(x) = g_1(f_1(x))$  for  $x \in X$ ,  $I = [0, 1]$  is the closed unit interval;  $I^m = I \times \cdots \times I$  ( $m$ -factors) is the unit  $m$ -cube;  $\bar{I}^m$  is the usual boundary set of  $I^m$ . The discussion is restricted to locally finite CW-complexes, for if  $Y$  is a locally finite CW-complex then  $Y \times Y$  is a CW-complex whose (closed)  $m$ -cells may be taken to be of the form  $E^m = E^p \times E^q$  ( $p+q=m$ ;  $E^p, E^q$  are, respectively,  $p$ -, and  $q$ -cells of  $Y$ ) [6]. The characteristic map of  $E^m$  is  $e^m = e^p \times e^q$ ;  $I^m \rightarrow E^m$  where  $e^p, e^q$  are character-

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istic maps of  $E^p, E^q$  and  $I^m$  is identified with  $I^p \times I^q$ . Given cochains  $c^p \in C^p(Y; G_1), c^q \in C^q(Y; G_2)$  and a pairing  $\xi: G_1 \otimes G_2 \rightarrow G$  we form a cochain  $c^p \times c^q \in C^{p+q}(Y \times Y; G)$  whose value on a cell  $E^p \times E^q$  is  $c^p \times c^q(E^p \times E^q) = \xi(c^p(E^p) \otimes c^q(E^q))$  and which is zero elsewhere. If  $c^p, c^q$  are cocycles then  $c^p \times c^q$  is, and the corresponding cohomology classes are  $[c^p], [c^q], [c^p] \times [c^q] = [c^p \times c^q]$ . If  $X$  is a CW-complex, its  $m$ -skeleton will be designated by  $X^m$ . The map  $\phi: Y \vee Y \rightarrow Y$ , presented in the introduction, will be called the *folding map* of  $Y$ . If  $f: A \rightarrow B$  is a map, its homotopy class will be designated by  $\{f\}$ .

If the folding map has been extended over  $Y \vee Y \cup (Y \times Y)^m$  then the obstruction to extending over the  $(m+1)$ -skeleton is in the cohomology group  $H^{m+1}(Y \times Y, Y \vee Y; \pi_m(Y))$ .

In case  $Y$  is  $(n-1)$ -connected,  $(Y \times Y, Y \vee Y)$  will be  $(2n-1)$ -connected.

**PROPOSITION 1.** *The obstruction in dimension  $2n$  to extending the folding map is*

$$d^n \times d^n \in H^{2n}(Y \times Y, Y \vee Y; \pi_{2n-1}(Y))$$

where  $d^n \in H^n(Y, y; \pi_n(Y))$  is the characteristic class for  $Y$  and the pairing of  $\pi_n(Y) \otimes \pi_n(Y)$  into  $\pi_{2n-1}(Y)$  is the J. H. C. Whitehead bracket product.

**PROOF.** We may replace  $Y$  by a space whose  $(n-1)$ -skeleton is a point (see §3). Then the  $(2n-1)$ -skeleton of  $Y \times Y$  is contained in  $Y \vee Y$  so that the obstruction cocycle to extending  $\phi$  is given by:

$$c^{2n}(E_1^n \times E_2^n) = \{ \phi \circ (e_1^n \times e_2^n) \mid I^{2n} \}$$

where  $E_1^n, E_2^n$  are cells of  $Y$  and  $e_1^n, e_2^n$  their characteristic maps. Note that

$$\{ \phi \circ (e_1^n \times e_2^n) \mid I^{2n} \} = [ \{ e_1^n \}, \{ e_2^n \} ]$$

with  $\{ e_1^n \}, \{ e_2^n \}$  regarded as elements of  $\pi_n(Y)$  [1]. But  $d_n$  is the class of the cocycle  $c^n$  given by

$$c^n(E^n) = \{ e^n \}.$$

Thus  $c^{2n} = c^n \times c^n$ .

**COROLLARY 2.**  $c^{2n} = 0$  if and only if  $[\alpha, \beta] = 0$  for  $\alpha, \beta \in \pi_n(Y)$ .

**PROOF.** Suppose  $c^{2n} = 0$ . Since  $H_n(Y) \approx \pi_n(Y)$  the characteristic maps of  $n$ -cells generate  $\pi_n(Y)$ . Thus  $[\alpha, \beta] = 0$  when  $\alpha, \beta$  are in this set of generators; hence  $[\alpha', \beta'] = 0$  for all  $\alpha', \beta' \in \pi_n(Y)$ . The converse is trivial.

On the other hand, if  $Y$  is an  $H$ -space then all bracket products vanish. This raises the question: if  $Y$  is a CW-complex and  $[\pi_p(Y), \pi_q(Y)] = 0$  for  $p, q > 0$ , is  $Y$  an  $H$ -space? The answer is negative, and we present a counter-example:

We construct a CW-complex,  $K$ , by specifying its  $m$ -skeleta,  $K^m$ . Let  $n > 2$  be an integer and  $p$  an odd prime. The  $2n$ -skeleton of  $K$  is taken to be the  $2n$ -skeleton of a CW-complex which is an Eilenberg-MacLane space of type  $(Z_p, n)$ , where  $Z_p$  is the integers mod  $p$ . There are no cells in dimension  $2n + 1$  ( $K^{2n+1} = K^{2n}$ ) and in higher dimensions, cells are appended so that  $\pi_i(K) = 0$  for  $i > 2n$  [7].

This creates a space whose homotopy groups are trivial except in dimensions  $n$  and  $2n$ . Thus all bracket products vanish. Also note that  $H^n(K; Z_p) \approx H^{n+1}(K; Z_p) \approx Z_p$  since the cohomology groups in these dimensions are those of a space of type  $(Z_p, n)$ . The  $(2n + 1)$ -cohomology group is zero, since there are no  $(2n + 1)$ -cells.

Now,  $K$  is not an  $H$ -space, for if it were, its cohomology ring would be a Hopf algebra and the cup product of an element of  $H^n(K; Z_p)$  with an element of  $H^{n+1}(K; Z_p)$  would be nonzero, contradicting  $H^{2n+1}(K; Z_p) = 0$  [2].

**2. The main result.** Let  $Y$  be a 1-connected CW-complex, and suppose that the folding map has been extended to  $\phi: Y \vee Y \cup (Y \times Y)^m \rightarrow Y$ . Let  $X$  be a CW-complex consisting of  $Y$  united with  $i$ -cells,  $E^i$ , ( $i > m$ ) such that  $\pi_i(X) = 0$  for  $i \geq m$ . Note that below dimension  $m$ , the inclusion map induces isomorphisms of the homotopy groups of  $X$  and  $Y$ .

**PROPOSITION 3.**  $X$  is an  $H$ -space.

**PROOF.** The  $m$ -skeleta of  $X$  and  $Y$  are the same, so that  $(X \times X)^m = (Y \times Y)^m$ . Thus  $\phi$  provides an extension  $\psi': X \vee X \cup (X \times X)^m \rightarrow X$ . But  $H^{i+1}(X \times X, X \vee X; \pi_i(X)) = 0$  for  $i > m$ , whence all obstructions to extending  $\psi'$  vanish. One such extension, let us call it  $\psi: X \times X \rightarrow X$ , is chosen for the structure map of  $X$ .

The condition  $\pi_i(X) = 0$  for  $i \geq m$  also implies that any two extensions of  $\psi'$  will be homotopic.

In the diagram below,  $i_1^*, \dots, i_5^*$  are homomorphisms induced by the appropriate inclusion maps:  $\psi^*, \psi_1^*$  are induced by  $\psi$ . The coefficient group for each of these cohomology groups is  $\pi_m(Y)$ . This symbol has been omitted to save space. It is well known that  $i_1^*$  sends  $H^{m+1}(X \times X, X \vee X; \pi_m(Y))$  isomorphically onto a direct summand of  $H^{m+1}(X \times X; \pi_m(Y))$ . This direct sum decomposition induces the homomorphism  $\rho$ . The composition  $\rho \circ i_1^*$  is the identity automorphism of  $H^{m+1}(X \times X, X \vee X; \pi_m(Y))$ , and the kernel of  $\rho$  is essen-

tially  $H^{m+1}(X \vee X; \pi_m(Y))$ . The diagram is commutative.

$$\begin{array}{ccccc}
 & & H^{m+1}(X + X, X \vee X) & & \\
 & \swarrow i_1^* & \nearrow \rho & \searrow i_2^* & \\
 H^{m+1}(X \times X) & \xleftarrow{i_3^*} & H^{m+1}(X \times X, Y \vee Y) & \xrightarrow{i_4^*} & H^{m+1}(Y \times Y, Y \vee Y) \\
 \psi_* \uparrow & & \uparrow \psi_1^* & & \\
 H^{m+1}(X) & \xleftarrow{i_5^*} & H^{m+1}(X, Y) & & 
 \end{array}$$

Let  $k' \in H^{m+1}(X, Y; \pi_m(Y))$  be the first obstruction to retracting  $X$  onto  $Y$ , and  $k = i_5^* k' \in H^{m+1}(X; \pi_m(Y))$ . The maps  $p_1, p_2: X \times X \rightarrow X$  are the projections  $p_i(x_1, x_2) = x_i$  for  $x_i \in X, i = 1, 2$ .

PROPOSITION 4. Let  $\gamma \in H^{m+1}(Y \times Y, Y \vee Y; \pi_m(Y))$  be the class of the obstruction to extending  $\phi$  to  $(Y \times Y)^{m+1} \cup Y \vee Y$ . Then

$$i_2^* \rho(\psi^* - p_1^* - p_2^*)k = \gamma.$$

PROOF. The cohomology class  $\psi_1^* k'$  is the first obstruction to extending  $(\psi|Y \vee Y \cup (X \times X)^m) = (\phi|Y \vee Y \cup (Y \times Y)^m)$  to  $X \times X$  and hence  $i_4^* \psi_1^* k' = \gamma$ . On the other hand,  $i_3^* \psi_1^* k' = \psi^* i_5^* k' = \psi^* k$ , and so  $i_2^* \rho \psi^* k = i_2^* \rho i_3^* \psi_1^* k' = i_4^* \psi_1^* k' = \gamma$ . Finally,  $\rho \circ (p_1^* + p_2^*)$  is the trivial homomorphism, whence  $i_2^* \rho(\psi^* - p_1^* - p_2^*)k = i_2^* \rho \psi^* k = \gamma$ .

If  $W$  is an  $H$ -space with  $\psi, p_1, p_2: W \times W \rightarrow W$  the structure map and the two projections respectively, then a cohomology class,  $u$ , is called primitive whenever  $(\psi^* - p_1^* - p_2^*)u = 0$ .

THEOREM 5. The obstruction class,  $\gamma$ , vanishes if and only if  $k$  is primitive.

PROOF. We already have  $i_2^* \rho(\psi^* - p_1^* - p_2^*)k = \gamma$  so that if  $k$  is primitive,  $\gamma = 0$ . To obtain the converse, we first note that  $i_2^*$  is an isomorphism since the inclusion map of  $Y$  in  $X$  induces isomorphisms  $H^i(X) \approx H^i(Y)$  for  $i \leq m$ . But the image of  $(\psi^* - p_1^* - p_2^*)k$  in  $H^{m+1}(X \vee X; \pi_m(Y))$  is zero, since  $(\psi|X \vee X)^* = (p_1|X \vee X)^* + (p_2|X \vee X)^*$ . Thus  $(\psi^* - p_1^* - p_2^*)k \in i_1^* H^{m+1}(X \times X, X \vee X; \pi_m(Y))$  from which it follows that  $\rho(\psi^* - p_1^* - p_2^*)k = 0$  implies  $(\psi^* - p_1^* - p_2^*)k = 0$ . This completes the proof.

Note that  $k$  is essentially the  $(m+1)$ -Postnikov invariant of  $Y$ . The theorem fails to provide a decisive victory over the problem of characterizing  $H$ -spaces which are CW-complexes, inasmuch as it depends upon choosing a particular extension,  $\psi$ , in each dimension. However, for sufficiently simple spaces the problem can be solved:

THEOREM 6. Suppose  $Y$  is a CW-complex which has only two non-trivial homotopy groups,  $\pi_n(Y)$  and  $\pi_m(Y)$ , with  $1 < n < m$ . Then  $Y$  is

an  $H$ -space if and only if the Eilenberg-MacLane  $k$ -invariant of  $Y$  is primitive.

PROOF. The only nontrivial obstruction to extending the folding map is in  $H^{m+1}(Y \times Y, Y \vee Y; \pi_m(Y))$ . Thus the space  $X$  is of type  $(\pi_n(Y), n)$  and  $k$  may be identified with the  $k$ -invariant,  $k_n^{m+1}(Y)$ . The result now follows from Theorem 4.

It may now be seen that the example of §1 was obtained by constructing a space with nonprimitive  $k$ -invariant.

In §3 it is shown that given abelian groups  $\pi_n, \pi_m$  and an element  $k \in H^{m+1}(\pi_n, n; \pi_m)$ , there is a space  $Y$  (which may be taken to be a CW-complex) such that  $\pi_i(Y) = 0$  for  $i \neq n, m$ ,  $\pi_n(Y) = \pi_n$ ,  $\pi_m(Y) = \pi_m$  and  $k_n^{m+1}(Y) = k$ . Any two such CW-complexes are of the same homotopy type. This observation and Theorem 5 give a classification of CW-complexes which admit  $H$ -structures and have only two non-vanishing homotopy groups.

Theorem 5 and Proposition 1 (§1) may be combined to yield a result about the Whitehead bracket product. Suppose  $\pi_i(Y) = 0$  for  $0 \leq i < n$  and  $n < i < 2n - 1$ . Let  $\pi_n = \pi_n(Y)$  and  $\pi_{2n-1} = \pi_{2n-1}(Y)$ ;  $\pi_n \otimes \pi_n, \pi_n \oplus \pi_n$  designate respectively the tensor product and the direct sum of  $\pi_n$  with itself. Three homomorphisms,  $\bar{\psi}, \bar{p}_1, \bar{p}_2: \pi_n \oplus \pi_n \rightarrow \pi_n$  are defined by  $\bar{\psi}(\alpha, \beta) = \alpha + \beta, \bar{p}_1(\alpha, \beta) = \alpha, \bar{p}_2(\alpha, \beta) = \beta$  for  $\alpha, \beta \in \pi_n$ .

PROPOSITION 7. *The cohomology class  $(\bar{\psi}^* - \bar{p}_1^* - \bar{p}_2^*)k_n^{2n}(Y) \in H^{2n}(\pi_n \oplus \pi_n, n; \pi_{2n-1})$  defines a homomorphism  $W: \pi_n \otimes \pi_n \rightarrow \pi_{2n-1}$  such that  $W(\alpha \otimes \beta) = [\alpha, \beta]$  for  $\alpha, \beta \in \pi_n$ .*

PROOF. Consider the diagram,

$$\begin{array}{ccc}
 \text{Hom} \{ \pi_n \otimes \pi_n; \pi_{2n-1} \} & \xleftarrow{\theta_1} & H^{2n}(\pi_n \oplus \pi_n, n; \pi_{2n-1}) \\
 \uparrow & & \downarrow \kappa^* \\
 & & H^{2n}(X \times X; \pi_{2n-1}) \\
 & & \uparrow i_1^* \circ (i_2^*)^{-1} \\
 \text{Hom} \{ H_n(Y) \otimes H_n(Y); \pi_{2n-1} \} & \xleftarrow[\theta_2]{} & H^{2n}(Y \times Y, Y \vee Y; \pi_{2n-1})
 \end{array}$$

The space  $X$  is of type  $(\pi_n, n)$ , so that the natural chain maps from the cell complex of  $X$  into the singular complex of  $X$  and thence into  $K(\pi_n, n)$  induces a chain map,  $\kappa$ , from the cell complex of  $X \times X$  into  $K(\pi_n \oplus \pi_n, n)$ . This last induces the isomorphism  $\kappa^*$ . The homomorphisms  $\bar{\psi}, \bar{p}_1, \bar{p}_2$  are algebraic analogues of  $\psi, p_1, p_2$ . In particular,  $\kappa^*(\bar{\psi}^* - \bar{p}_1^* - \bar{p}_2^*)k_n^{2n}(Y) = (\psi^* - p_1^* - p_2^*)k$ . The homomorphisms  $i_1^*, i_2^*$

are as defined in Theorem 5;  $\eta: \pi_n \rightarrow H_n(Y)$  is the Hurewicz isomorphism and  $(\eta \otimes \eta)^*$  is the induced isomorphism of the Hom groups. The Kunneth formula yields the homomorphisms  $\theta_1, \theta_2$ ;  $\theta_2$  is an isomorphism. Note that  $\theta_1 \circ (\kappa^*)^{-1} \circ i_1^* \circ (i_2^*)^{-1} \circ \theta_2^{-1} \circ ((\eta \otimes \eta)^*)^{-1}$  is the identity automorphism of  $\text{Hom} \{ \pi_n \otimes \pi_n; \pi_{2n-1} \}$  onto itself.

Recall that  $d_n \in H^n(Y; \pi_n)$  is the basic cohomology class of  $Y$ . Thus the image of  $d_n$  in  $\text{Hom} \{ H_n(Y); \pi_n \}$  is  $\eta^{-1}$  and  $\theta_1(W \circ (\eta^{-1} \otimes \eta^{-1})) = d_n \times d_n = \gamma$ . We now have,

$$\begin{aligned} W &= \theta_1 \circ (\kappa^*)^{-1} \circ i_1^* \circ (i_2^*)^{-1} \circ \theta_2^{-1} \circ ((\eta \otimes \eta)^*)^{-1}(W) \\ &= \theta_1 \circ (\kappa^*)^{-1} \circ i_1^* \circ (i_2^*)^{-1}(\gamma) \\ &= \theta_1 \circ (\Psi^* - \tilde{p}_1^* - \tilde{p}_2^*) k_n^{2n}(Y). \end{aligned}$$

PROPOSITION 8. *Suppose  $Y$  is a CW-complex with only two nonvanishing homotopy groups,  $\pi_n = \pi_n(Y)$  and  $\pi_m = \pi_m(Y)$ . Then there is a space of loops  $\Omega$  having the same homotopy groups and  $k$ -invariant as  $Y$  if and only if  $k_n^{m+1}(Y)$  is the suspension of an element*

$$k^{m+2} \in H^{m+2}(\pi_n, n + 1; \pi_m).$$

PROOF. Let  $S$  be the suspension homomorphism and suppose  $k_n^{m+1}(Y) = S k^{m+2}$ . Let  $W$  be a space such that  $\pi_i(W) = 0$  ( $i \neq n + 1, m + 1$ ),  $\pi_{n+1}(W) = \pi_n$ ,  $\pi_{m+1}(W) = \pi_m$ ,  $k_{n+1}^{m+2}(W) = k^{m+2}$ . Then  $\Omega =$  the space of loops on  $W$  is the desired space. The converse is immediate.

J. C. Moore has demonstrated (unpublished) that if  $\alpha \in H^{m+1}(\pi, n; G)$  is primitive then  $\alpha$  is the suspension of an element of  $H^{m+2}(\pi, n + 1; G)$ . Thus all  $H$ -spaces of the type under discussion are essentially spaces of loops.

3. Let  $X$  be a given 0-connected space,  $x_0 \in X$ . We construct a CW-complex,  $K(X)$ , whose 0-skeleton is a point  $k_0 \in K(X)$ , and a map  $f(X): K(X) \rightarrow X$  such that  $f(X)$  induces isomorphisms  $f(X)_\#: \pi_i(K(X), k_0) \rightarrow \pi_i(X, x_0)$  for  $i \geq 0$ . This is done by specifying the  $n$ -skeleta  $K^n$  of  $K(X)$  and maps  $f_n: K^n \rightarrow X$ .

The 0-skeleton consists of one cell  $E^0 = k_0$ . Suppose  $K^n$  and  $f_n: K^n \rightarrow X$  are constructed such that the induced homomorphism

$$f_{n\#}^{(i)}: \pi_i(K^n, k_0) \rightarrow \pi_i(X, x_0)$$

is an isomorphism for  $i < n$  and onto for  $i = n$ . Let  $A_{n+1}$  be the kernel of  $f_{n\#}^{(n)}$  and  $B_{n+1} \subset \pi_{n+1}(X, x_0)$  a set of generators of  $\pi_{n+1}(X, x_0)$ . Append cells  $E_\alpha^{n+1}$  ( $\alpha \in A_{n+1}$ ), so that if  $e_\alpha^{n+1}$  is the characteristic map of  $E_\alpha^{n+1}$  then  $(e_\alpha^{n+1} | \dot{I}^{n+1}) \in \alpha$ , and cells  $E_\beta^{n+1}$  ( $\beta \in B_{n+1}$ ) with  $e_\beta^{n+1} (\dot{I}^{n+1}) = k_0$ . The map  $f_n$  can be extended over cells  $E_\alpha^{n+1}$  ( $\alpha \in A_{n+1}$ ) since

$f_n|e_\alpha^{n+1}(\dot{I}^{n+1})$  is null-homotopic;  $f_{n+1}|E_\beta^{n+1}(\beta \in B_{n+1})$  is determined by  $f_{n+1} \circ e_\beta^{n+1} \in \beta$ . Then

$$f_{n+1\sharp}^{(i)}: \pi_i(K^{n+1}, k_0) \rightarrow \pi_i(X, x_0)$$

is an isomorphism for  $i < n + 1$  and onto for  $i = n + 1$ .

The complex  $K(X)$  is  $\bigcup_{n=0}^\infty K^n$ , with the topology:  $C$  is closed in  $K(X)$  if and only if  $C \cap K^n$  is closed in  $K^n$  for each  $n$ . The map  $f(X): K(X) \rightarrow X$  is given by  $(f(X)|K^n) = f_n$ . Note that this is a modification of a construction by J. H. C. Whitehead [7].

PROPOSITION 9. *If  $\pi_n, \pi_m$  are abelian groups ( $n < m$ ) and  $k \in H^{m+1}(\pi_n, n; \pi_m)$  then there is a CW-complex  $K$  such that  $\pi_i(K) = 0$  for  $i \neq n, m$ ,  $\pi_n(K) = \pi_n, \pi_m(K) = \pi_m, k^m = k$ .*

PROOF. Let  $E$  be the space of paths in the Eilenberg-MacLane space  $K(\pi_m, m + 1)$  terminating in some point  $y_0 \in K(\pi_m, m + 1)$  with fibre map  $p_1: E \rightarrow K(\pi_m, m + 1)$  and fibre  $K(\pi_m, m)$ . If  $d \in H^{m+1}(\pi_m, m + 1; \pi_m)$  is the basic cohomology class, then there is a map  $f: K(\pi_n, n) \rightarrow K(\pi_m, m + 1)$  such that  $f^*(d) = k \in H^{m+1}(\pi_n, n; \pi_m)$ . Note that  $K(\pi_m, m + 1), K(\pi_n, n)$  may be chosen to be CW-complexes and  $f$  cellular. The map  $f$  induces a space  $X$  and maps  $p_2, F$  such that the diagram

$$\begin{array}{ccc} & F & \\ X & \longrightarrow & E \\ p_2 \downarrow & & \downarrow p_1 \\ K(\pi_n, n) & \longrightarrow & K(\pi_m, m + 1) \end{array}$$

is commutative and  $X$  is a fibre space over  $K(\pi_n, n)$  with fibre map  $p_2$  and fibre  $K(\pi_m, m)$ . From the homotopy sequence of the fibre map  $p_2$  we see that  $\pi_i(X) = 0$  for  $i \neq n, m, \pi_n(X) = \pi_n, \pi_m(X) = \pi_m$ .

We know that there is a map,  $j$ , of the  $m$ -skeleton of  $K(\pi_n, n)$  into  $X$  such that  $p_2 \circ j$  is the identity, and that the obstruction to extending  $j$  is  $k_n^{m+1}$ . If  $E^{m+1}$  is an  $(m + 1)$ -cell of  $K(\pi_n, n)$  then its characteristic map  $e^{m+1}$  (considered as a null-homotopy of  $(e^{m+1}| \dot{I}^{m+1})$ ) can be lifted to a map  $g: \dot{I}^{m+1} \times I \rightarrow X$  with  $(g| \dot{I}^{m+1} \times 0) = j \circ (e^{m+1}| \dot{I}^{m+1})$  and  $g' = (g| \dot{I}^{m+1} \times 1): \dot{I}^{m+1} \rightarrow K(\pi_m, m)$ . If  $\partial$  is the boundary homomorphism of the homotopy sequence of  $p_1$  and  $F' = (F|K(\pi_m, m))$  then

$$\partial^{-1}F'_\sharp\{g'\} = \{f \circ e^{m+1}\} \in \pi_{m+1}(K(\pi_m, m + 1)).$$

But  $f^*d(E^{m+1}) = \{f \circ e^{m+1}\}$ . Thus there is an isomorphism of  $H^{m+1}(\pi_n, n; \pi_m(X))$  onto  $H^{m+1}(\pi_n, n; \pi_{m+1}(K(\pi_m, m + 1)))$  carrying

$k_n^{m+1}$  into  $k = f*d$ . The desired CW-complex is then obtained as in the beginning of this section.

This proof is the obvious generalization of one given by Thom [5].

#### BIBLIOGRAPHY

1. A. L. Blakers and W. S. Massey, *Products in homotopy theory*, Ann. of Math. vol. 58 (1954) pp. 295–324.
2. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. vol. 57 (1953) pp. 115–207.
3. P. J. Hilton, *An introduction to homotopy theory*, Cambridge University Press, no. 43, 1953.
4. M. M. Postnikov, *On the homotopy type of polyhedra*, Doklady Akademii Nauk SSSR. (N.S.) vol. 76 (1951) pp. 789–791.
5. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. vol. 28 (1954) pp. 17–86.
6. J. H. C. Whitehead, *Combinatorial homotopy I*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 213–245.
7. ———, *On the realizability of homotopy groups*, Ann. of Math. vol. 50 (1949) pp. 261–263.

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