CLASSES OF MAXIMUM NUMBERS ASSOCIATED WITH TWO SYMMETRIC EQUATIONS IN N RECIPROCALS\textsuperscript{1}

H. A. SIMMONS

1. Introduction. As in papers \[l\]\textsuperscript{2} and [2], we let \(\sum_{i,j} (1/x)\) stand for the elementary symmetric function of the \(j\)th order of the \(i\) reciprocals \((1/x_i), (p = 1, \cdots, i; i > 0)\) with

\[
\sum_{i,j} (1/x) \equiv 0 \text{ when } i < j \text{ or } j < 0,
\]

\[
\equiv 1 \text{ when } j = 0
\]

(\(\sum_{i,j} (x)\) having a similar meaning for the \(x_p\) themselves).

In §§1–4, we consider the equation

\[
\sum_{n,r} (1/x) + L_{r+1} \sum_{n,r+1} (1/x) + \cdots + L_s \sum_{n,s} (1/x) = b/a,
\]

\[a = (c + 1)b - 1,
\]

in which \(r, s, n\) are positive integers with \(r < s \leq n\); each \(L_p, (p = r+1, \cdots, s)\), is a non-negative integer; and \(b, c\) are arbitrary positive integers.

In §5, we shall state results for another equation which follow from the procedure of §§1–4.

For convenience, we define

\[
\phi_p(1/x) = \sum_{p,r} (1/x) + L_{r+1} \sum_{p,r+1} (1/x) + \cdots + L_s \sum_{p,s} (1/x),
\]

\[(r \leq p \leq n).\]

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\textsuperscript{1} Due to ingenuity of the referee, we have changed our method of reasoning about this material; a different use of relations (6) here has caused us to modify our transformation procedure. On account of this, we have replaced our former “E-solutions” by much more general “admissible solutions.” Consequently, we have generalized the result in our first manuscript to Theorem 1 here, and we have done so relatively briefly. Furthermore, we have identified existentially every admissible solution of equation (1) that we would use in passing from a given admissible solution \(x\) of (1), \(x\) different from the Kellogg solution \(w\) in (5), to \(w\).

The material of §4 here would not have occurred to us; it is entirely due to the referee, and it is largely in his wording. We are extremely grateful for his assistance, so much so that we offered to share the title of this paper with him.

\textsuperscript{2} Numbers in square brackets refer to papers whose titles appear in the list of references at the end of this article.
Consequently, (1) is expressed by $\phi_n(1/x) = b/a$ with $a$ as in (1).

Hence, using the notation

$$N_p(x) = \sum_{p, p-r+1} (x) + L_{r+1} \sum_{p, p-r} (x) + \cdots + L_n \sum_{p, p-r+1} (x),$$

$(p = r, \ldots, n - 1)$,

and (2), we also write, for convenience,

$$\phi_p(1/x) = \frac{1}{x_1 \cdots x_r} + \sum_{t=r}^{p-1} \frac{1}{x_{t+1}} \frac{N_t(x)}{x_1 \cdots x_t} \quad (r < p \leq n).$$

By definition, the Kellogg solution [3] of (1) is obtained by taking the variables $x_1, x_2, \cdots, x_{n-1}$ in this order as small as possible (among positive integers). In obtaining this solution, say $w$, one would have $w_p = 1$, $(p = 1, \cdots, r - 1)$, and $w_r = c + 1$. Also since the term summed in (4) equals

$$\frac{[aN_t(x) + 1]}{x_{t+1}(ax_1 \cdots x_t)},$$

which reduces to $(ax_1 \cdots x_t)^{-1} - (ax_1 \cdots x_{t+1})^{-1}$ when $x_{t+1} = aN_t(x)$ +1—the Kellogg choice for $x_{t+1}$ after the first $t$'s have been selected by his method—one readily completes the. Kellogg solution $x = w$, namely:

$$w_p = 1, \quad (p = 1, \cdots, r - 1), \quad w_r = c + 1,$$

$$w_{p+1} = aN_p(w) + 1, \quad (p = r, \cdots, n - 2), \quad w_n = aN_{n-1}(w).$$

Henceforth, we have called any solution of (1) in which

(i) $x_1 \leq x_2 \leq \cdots \leq x_n$;

(ii) the $x_p$, $(p = 1, \cdots, n - 1)$, are positive integers

an $E$-solution of (1), and we have considered such solutions only. However, the procedure to be used here reveals that we can obtain greater generality by merely requiring that the elements $x_j$ in each solution $x$ of (1) be real numbers $\geq 1$, satisfying (i), and such that

$$\phi_p(1/x) \leq b/a - 1/(ax_1 \cdots x_p), \quad (p = r, \cdots, n - 1).$$

Henceforth such solutions will be called admissible solutions, and we shall confine our attention to these.

Reasoning used in [1] shows that when $x$ is an $E$-solution of (1), (6) holds for each indicated value of $p$. Furthermore, one readily finds that when $x = w$, the equality sign applies in (6) for each value of $p$. Being an $E$-solution of (1), $w$ is admissible.

Now let $P(x) \equiv P(x_1, x_2, \cdots, x_n)$ be any nonconstant, symmetric
The result which we are to prove relative to (1) is as follows.

**Theorem 1.** If \( x \) is an admissible solution and \( x \neq w \), then \( x_n < w_n \) and \( P(x) < P(w) \).

2. **Lemmas that we use in proving Theorem 1.** The following lemma is essentially Lemma 1a of [1].

**Lemma 1.** Let \( Q(1/x) \) stand for a symmetric polynomial in the \( u \) reciprocals \( (1/x_p), (p = 1, \cdots, u; u > 1) \), which is not a mere constant and contains at least one positive, and no negative, coefficient; with \( i \) and \( j \) equal to distinct positive integers each less than or equal to \( u \), let \( x_i, x_j, \alpha, \beta \) be positive numbers with \( \alpha < x_i \leq x_j \); and suppose that the expression which is obtained by replacing in \( Q(1/x) \) the numbers \( x_i, x_j \) by \( (x_i - \alpha), (x_j + \beta) \), respectively, equals \( Q(1/x) \); then

\[
(7) \quad x_i x_j \leq (x_i - \alpha)(x_j + \beta), \quad x_i + x_j < (x_i - \alpha)^h + (x_j + \beta)^h,
\]

where \( h \) is any positive integer. Furthermore, the equality sign holds in (7) if, and only if, \( Q(1/x) \) is a polynomial in \( (x_1 \cdots x_u)^{-1} \).

**Lemma 2.** If \( x \) is an admissible solution for which the equality sign applies in (6) for \( p = k, (r \leq k < n) \), then \( x_{k+1} > x_k \) except when (1) is \( x_1^{-1} + x_2^{-1} = 1 \).

**Proof.** By hypothesis and our definition of \( \phi_p(1/x) \), we have

\[
(ax_1 \cdots x_k)^{-1} = b/a - \phi_k(1/x) = \phi_n(1/x) - \phi_k(1/x) \\
\geq \phi_{k+1}(1/x) - \phi_k(1/x) \\
\geq x_{k+1}^{-1} \sum_{k, r=1}^{k} (1/x),
\]

so that

\[
x_{k+1} \geq a \sum_{k, k-r=1}^{k} (x) \geq ax_k \geq x_k
\]

with the inequality sign holding throughout except when \( n = k+1 = 2 \) and \( r = 1 = a \); that is, except in the special case indicated in the statement of Lemma 2.

**Terminology.** Henceforth, if for an admissible solution the sign \( < \) holds in (6) for \( p = k, (r \leq k < n) \), to reduce the \( k \)th element broadly until the equality sign holds in (6) for \( p = k \) will mean that:

(1°) we reduce the \( k \)th element only until the equality is reached if this does not require the \( k \)th element to be less than the \((k-1)\)th;
(2°) if the $k$th element is equal to one or more elements with smaller subscript, or becomes so while being reduced before the equality for $p = k$ in (6) is reached, then each such element is to be kept equal to the $k$th in the rest of its reduction.

**Lemma 3.** If $x$ is an admissible solution for which the inequality sign holds in (6) for $p = m$, $(r \leq m \leq n - 1)$, and if we reduce the $m$th element of $x$ broadly until the equality sign holds in (6) for $p = m$, the new set $(X_1, \ldots, X_m)$ obtained is such that

\[ X_1 \leq X_2 \leq \cdots \leq X_m; \]

\[ \phi_p(1/X) \leq b/a - (aX_1 \cdots X_p)^{-1}, \quad (r \leq p \leq m). \]

**Proof.** By (1°) and (2°) above, reducing the $m$th element broadly until the equality sign holds in (9) for $p = m$ yields new elements $X_u$, $(u = 1, \ldots, m)$, which satisfy (8).

Relative to (9), by hypothesis the equality sign holds in it when $p = m$; so if there is just one case in (9), namely $r = m$, then (9) is true. Suppose $r < m$. Then reducing $x_m$ only would not change the value of $\phi_p(1/x)$, $(r \leq p < m)$. Also if at some stage of our broad reduction of the $m$th element, the numbers $x_p$, $(p = 1, \ldots, m)$ are $x'_p$ with $x'_m = x'_{m-1}$, and if an admissible solution $x'$ is completed by using $x'$ instead of $x$ in equations (11) below, then Lemma 2 guarantees that we could not have

\[ \phi_{m-1}(1/x') = b/a - (ax'_1 \cdots x'_{m-1})^{-1}; \]

so, in the reduction while $x'_m = x'_{m-1}$, we would have $\phi_{m-1}(1/x') < b/a - (ax'_1 \cdots x'_{m-1})^{-1}$. Similarly, we could not have simultaneously

\[ x_m' = x_p', \quad \phi_p(1/x') = b/a - (ax'_1 \cdots x_p')^{-1}, \quad (r \leq p < m - 1), \]

when there is at least one value of $p$ satisfying the indicated conditions. Reasoning similarly about cases in which $x'_u = x'_v$, where $u$ and $v$ are distinct values of $p$, $(r \leq p \leq m - 1)$, completes the proof that the set $X_1, \ldots, X_m$ which is obtained from $x_1, \ldots, x_m$ by decreasing $x_m$ broadly satisfies (9).

Reasoning used in §1 about the term summed in (4) yields the following lemma.

**Lemma 4.** If $x$ is an admissible solution for which the equality sign holds in (6) when $p = m$, $(r \leq m \leq n - 1)$, then the equations

\[ \phi_p(1/x) = b/a - (ax_1 \cdots x_p)^{-1}, \quad (p = m + 1, \ldots, n - 1), \]

hold if, and only if, the $x$'s with subscript larger than $m$ are as follows:
(11) \( x_p = aN_{p-1}(x) + 1, \quad (p = m + 1, \ldots, n - 1), \) \( x_n = aN_{n-1}(x), \)

where for \( m = n - 1 \) all equations in (10) are to be discarded, and all in (11) except its last equation.

3. Proof of Theorem 1. If \( x \) is an admissible solution for which the inequality sign holds in (6) for \( p = n - 1 \), keeping the left member of (1) equal to \( b/a \), we decrease \( x_{n-1} \) broadly until the equality sign holds in (6) for \( p = n - 1 \) and we increase \( x_n \) accordingly. The new set \( X \) obtained satisfies the case \( m = n - 1 \) of (8) and (9) with equality in the latter for \( p = n - 1 \); and \( X_n = aN_{n-1}(X) > x_n \). Obviously \( X \) is admissible. Furthermore, in transforming from \( x \) to \( X \), every element that was decreased was no larger than the single element \( x_n \) which was increased. Consequently, by Lemma 1, \( P(X) > P(x) \).

Next, if \( x \) is an admissible solution of (1) for which the inequality sign holds in (6) for \( p = k, \quad (r \leq k < n - 1) \), but not for \( p \) equal to any one of the numbers \( k + 1, \ldots, n - 1 \) (where this set is now not vacuous), so that, in particular for \( p = k + 1 \),

\[
\phi_{k+1}(1/x) + (ax_1 \cdots x_{k+1})^{-1} = b/a,
\]

then maintaining this equality, we decrease \( x_k \) broadly until the equality sign holds in (6) for \( p = k \), and we increase \( x_{k+1} \) accordingly. The first \( k + 1 \) elements of the new set \( X \) satisfy the case \( k + 1 = m \) of (8) and (9) with equality in (9) for \( p = k + 1 \). By Lemma 1, the product and the sum of the \( h \)th powers of the \( k + 1 \) elements \( X_1, \ldots, X_{k+1}, \) \( h \) any positive integer, exceed the product and sum of the \( h \)th powers, respectively, of the corresponding elements \( x_1, \ldots, x_{k+1} \). (We let \( X_p \) in \( X \) correspond to \( x_p \) in \( x \).) Furthermore, it is obvious that we can complete an admissible solution \( X \) of (1) by using \( X \) in the place of \( x \) in those cases of (11) which define the \( X_j (j = k+2, \ldots, n) \); also, from the nature of our transformation and Lemma 1, we know that \( N_p(X) > N_p(x) \) for \( p = k + 1, \ldots, n - 1 \). Consequently

\[
(12) \quad X_n > x_n; \quad P(X) > (Px), \quad \text{(Lemma 1)}.
\]

The two types of argument used in the last two paragraphs above guarantee that if \( x \) is an admissible solution for which the inequality sign holds in one or more cases of (6), we can transform \( x \) into another admissible solution \( X \) for which the equality sign holds in (6) for every \( p \). Furthermore, such argument shows that (12) holds. Consequently, if \( r = 1 \), so that \( X \) is necessarily \( w \), we are through.

Next, suppose that \( r > 1 \) and that \( x, (x \neq w) \), is an admissible solution for which the equality sign holds in (6) for \( p = r, \ldots, n - 1 \); now necessarily \( x_{r-1} > 1 \). Then, keeping
we decrease to 1 every $x_p$, ($p = 1, \cdots, r-1$) that exceeds 1, and we increase $x_r$ accordingly. The product of the first $r$ elements is left unchanged (= $c+1$), but, by Lemma 1, the sum of the $h$th powers of these elements is increased. Consequently, when we complete our admissible solution $X$ by writing [cf. (11)]

$$X_p = aN_{p-1}(X) + 1, \quad (p = r + 1, \cdots, n - 1), \quad X_n = aN_{n-1}(X),$$

no one of these elements will be less than its correspondent in $x$, and, since $n > r$, we shall surely have $X_n > x_n$ and $P(X) > P(x)$. Since $X$ is now necessarily $w$, our proof of Theorem 1 is complete.

4. Another proof of Theorem 1.

**Lemma 5.** The admissible solutions $x$—considered as points in the space of the variables $x_1, \cdots, x_n$ under our specified conditions (of admissibility)—form a closed, bounded region of $n$-space.

**Proof.** Suppose false. Then there exists a sequence of admissible solutions $x^{(1)}, x^{(2)}, \cdots$, such that $x_n^{(v)}$ tends to infinity as $v$ does so. Since $0 < b/a = \phi_n(1/x)$, we see that $x_n^{(v)}$ does not tend to infinity for some fixed $N$, $1 \leq N < n$. Without loss of generality, suppose $N$ maximal. Then we have

$$\lim_{v \to \infty} \phi_N(1/x^{(v)}) = \lim_{v \to \infty} \phi_n(1/x^{(v)}) = b/a.$$

Hence $\lim_{v \to \infty} (ax_1^{(v)} \cdots x_n^{(v)})^{-1} = 0$, which is a contradiction since

$$1 \leq x_1^{(v)} \leq \cdots \leq x_n^{(v)}.$$

Now to prove Theorem 1, it suffices to prove this: if $x$ is an admissible solution $\neq w$, there exists another admissible solution $y$ of (1) such that

$$y_n > x_n, \quad P(y) > P(x).$$

By reference to §3, we see that the first $X$ into which our procedure carried any admissible solution $x$ other than $w$ can be taken as $y$; and the proof is complete.

5. Another equation and our result for it. Consider the equation

$$(13) \quad L_r \sum_{n,r} (1/x) + L_{r+1} \sum_{n,r+1} (1/x) + \cdots + L_s \sum_{n,s} (1/x) = 1,$$

in which $L_r$ is a positive integer; each $L_i$, ($i = r+1, \cdots, s$), is a non-negative integer; and $r, s, n$ are positive integers with $r < s \leq n$. 
Equation (13) contains equations which are not included in (1), and our procedure of §§1–3 applies to (13). After a few definitions, we shall state our result for (13).

To obtain our new definitions of $\phi_p(1/x)$ and $N_p(x)$ from (2) and (3), respectively, merely multiply the first term in the right member of (2) and of (3) by $L_r$. Also we define any solution $x$ of (13) to be admissible if each $x_i \geq 1$, $(i = 1, \cdots, n)$, $x_1 \leq x_2 \leq \cdots \leq x_n$, and the case $a = b = 1$ of (6) holds.

By procedure heretofore described, one finds the Kellogg solution of (13) to be $x = W$, where

\begin{equation}
W_p = 1, \quad (p = 1, \cdots, r - 1), \quad W_r = L_r + 1, \\
W_{p+1} = N_p(W) + 1, \quad (p = r, \cdots, n - 2), \quad W_n = N_{n-1}(W).
\end{equation}

**Theorem 2.** If $x$ is an admissible solution of (13) other than $W$, of (14), then

\[ W_n > x_n, \quad P(W) > P(x). \]

**References**


Northwestern University