

A NOTE ON THE COMPOSITENESS OF NUMBERS

A. W. ADDISON

The "compositeness" of the number $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ is defined by $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. If the integers be partitioned into two classes E_0 and E_1 according to whether $\Omega(n) \equiv 0, 1 \pmod{2}$, and $E_0(x), E_1(x)$ be the corresponding counting functions, it follows that $E_i(x) = (x/2) + \text{error}$. The error is $O(x \exp[-a(\log x)^{1/2}])$ certainly, and on the Riemann hypothesis is $O(x^{1/2+\epsilon})$. This becomes evident when one considers $\zeta(2s)/\zeta(s)$, which is the generating function for $E_0(x) - E_1(x)$.

However, there is no "analogy" on the "error term" if the partitioning follow the residues of a number larger than 2, as we shall show. In fact, we shall establish the following

THEOREM. *If for any $q \geq 3$ we partition the integers into q classes $\{C_{q,i}\}$, ($i=0, 1, \dots, q-1$), according to whether $\Omega(n) \equiv 0, 1, \dots, q-1 \pmod{q}$ and let $C_{q,i}(x)$ be the corresponding counting functions, it follows that*

$$C_{q,i}(x) - x/q = \Omega_{\pm}(x/\log^r x),$$

($i=0, 1, \dots, q-1$), where $r = 1 - \cos(2\pi/q)$.

The leading term x/q , with error of $o(x)$, has already been established by several investigators [1; 2] who made use only of elementary (non "complex-variable") arguments.

Specifically, we shall here actually compute the remainder term for $q=3$. For larger values the computation is more complicated only with respect to notation, and we shall merely state the result.

Write $\omega_1 = \exp(2\pi i/3) = (-1/2) + i(3^{1/2}/2)$, and $\omega_2 = \omega_1^2$. Define

$$F(s) = \prod_p \left(1 - \frac{\omega_1}{p^s}\right)^{-1}$$

so that

$$C_{3,0}(x) + \omega_1 C_{3,1}(x) + \omega_2 C_{3,2}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s) \frac{x^s}{s} ds,$$

where of course Cauchy mean value is understood. We shall show that $F(s)$ has an algebraic singularity at $s=1$, and behaves somewhat like $(s-1)^{-\omega_1}$ in the neighborhood of that point. To this effect, expand

Received by the editors April 29, 1955 and, in revised form, March 15, 1956.

the logarithm of $F(s)$ and group the terms according to the coefficients $\omega_1, \omega_2, 1$. Then from

$$\sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns)$$

it will follow that $\log F(s) = \omega_1 \log \zeta(s) + \log g(s)$ where $\log g(s)$ is regular and bounded in absolute value for $\sigma \geq 1/2 + \epsilon$. So $F(s) = (\zeta(s))^{\omega_1} g(s)$; $g(s) \neq 0$ for $\sigma \geq 1/2 + \epsilon$. Write $f(s) = F(s)/s = (s-1)^{-\omega_1} h(s)$ and note that $h(s)$ is regular and bounded in absolute value in the neighborhood of $s=1$. Further, $h(s)$ does not vanish in this region. Thus

$$h(s) = k_0 + k_1(1-s) + \dots + k_n(1-s)^n + \dots \quad (k_0 \neq 0),$$

valid for $|1-s| \leq 1-a$.

Our problem is thus reduced to something similar to that of estimating the number of integers which are the sum of two squares [3]. Run the contour of integration from $2-i\infty$ to $1-a \geq 1/2 + \epsilon$, following the path $\sigma = 1 - (a/\log(e+|t|))$. From $1-a$, run to $1-\eta$, then on the circle of radius η about the point $s=1$, then from $1-\eta$, back to $1-a$, and from there to $2+i\infty$. Now this integral (except for the parts running along the real axis and about $s=1$) is $O(x \exp[-a(\log x)^{1/2}])$ certainly, and on the Riemann hypothesis is $O(x^{1/2+\epsilon})$. For the rest, consider the first term of our integral

$$\frac{k_0}{2\pi i} \int_{1-a}^{1-\eta} (s-1)^{-\omega_1} x^s ds.$$

Factor out $(-1)^{-\omega_1}$ from $(s-1)^{-\omega_1}$ and make the substitution $t = (1-s) \log x$. For $\eta = 0$ this is equal to

$$\begin{aligned} & \frac{k'_0 (-1)^{-\omega_1}}{2\pi i} \cdot \frac{x}{(\log x)^{1-\omega_1}} + O(x^{1-a+\epsilon}) \\ & = k''_0 \frac{x}{(\log x)^{3/2}} \exp\left(i \frac{3^{1/2}}{2} \log \log x\right) + O(x^{1-a+\epsilon}), \end{aligned}$$

where $k'_0, k''_0 \neq 0$.

The integral taken about the point $s=1$ goes to zero with η , but upon returning to the point $1-\eta$ a new factor $\neq 1$ now multiplies the integrand. Thus the integral returning along the real axis merely changes the constant k''_0 to, say, $m_0 \neq 0$.

Repeating for the other terms in $(1-s)^n$ in the series we obtain the asymptotic expansion

$$\frac{x}{(\log x)^{3/2}} \left\{ \exp \left(i \frac{3^{1/2}}{2} \log \log x \right) \right\} \left(m_0 + \frac{m_1}{\log x} + \cdots + \frac{m_n}{\log^n x} + \cdots \right).$$

To isolate the function $C_{3,0}(x)$, for example, write

$$G(s) = \prod_p \left(1 - \frac{\omega_2}{p^s} \right)^{-1}.$$

This will yield an asymptotic expansion identical with the above, except for a replacement of the constants m_j by \bar{m}_j (their complex conjugates) and a replacement of i by $-i$ in the exponent of e .

The function $G(s)$ is the generating function for $C_{3,0}(x) + \omega_2 C_{3,1}(x) + \omega_1 C_{3,2}(x)$. Thus $F(s) + G(s)$ is the generating function for $3C_{3,0}(x) - [x]$. Thus

$$3C_{3,0}(x) - x \sim \frac{x}{(\log x)^{3/2}} \cdot \left\{ \left(\cos \left(\frac{3^{1/2}}{2} \log \log x \right) \right) \left(a_0 + \frac{a_1}{\log x} + \cdots + \frac{a_n}{\log^n x} + \cdots \right) - \left(\sin \left(\frac{3^{1/2}}{2} \log \log x \right) \right) \left(b_0 + \frac{b_1}{\log x} + \cdots + \frac{b_n}{\log^n x} + \cdots \right) \right\}$$

where $a_j + ib_j = 2m_j \neq 0$.

For $q > 3$ the analogous computations yield

$$qC_{q,0}(x) - x \sim \sum_{j=1}^{[(q-1)/2]} \frac{x}{(\log x)^{r_j}} \left\{ (\cos (v_j \log \log x)) \left(a_{0j} + \frac{a_{1j}}{\log x} + \cdots \right) - (\sin (v_j \log \log x)) \left(b_{0j} + \frac{b_{1j}}{\log x} + \cdots \right) \right\}$$

where $r_j = 1 - \cos (2\pi j/q)$; $v_j = \sin (2\pi j/q)$; and at least one of the numbers a_{01}, b_{01} is different from zero. This establishes our theorem.

For completeness, we mention the other measure of "compositeness" in use. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, ($\alpha_i > 0$) we could also define $\Omega^*(n) = m$, that is to say the number of *distinct* primes dividing n , not counting multiplicity. It suffices to consider the generating functions

$$H(s) = \prod_p \left(1 + \frac{\omega_1}{p^s - 1} \right)$$

and

$$I(s) = \prod_p \left(1 + \frac{\omega_2}{p^s - 1} \right)$$

which will yield a similar formula for $q=3$. The general case likewise holds.

Finally we mention the "square-free" case. The square-frees are counted by the function $Q(x)$. Partition them into three classes $Q_{3,0}$, $Q_{3,1}$, $Q_{3,2}$, etc. Here it suffices to note that

$$\prod_p \left(1 + \frac{\omega_1}{p^s} \right) = \frac{F(s)}{G(2s)}$$

and

$$\prod_p \left(1 + \frac{\omega_2}{p^s} \right) = \frac{G(s)}{F(2s)}$$

where F and G are defined as before. This again leads to a similar formula, with $x/3$ replaced by $Q(x)/3$. The general case likewise holds.

REFERENCES

1. S. Selberg, *Zur Theorie der quadratfreien Zahlen*, Math. Zeit. vol. 44 (1938) pp. 306–318.
2. S. S. Pilla, *Generalisation of a theorem of Mangoldt*, Proceedings of the Indian Academy of Sciences, Sect. A, vol. 11 (1940) pp. 13–20.
3. E. Landau, *Primzahlen*, Chelsea reprint, p. 645 ff.

NEW YORK UNIVERSITY