The "compositeness" of the number \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_m^{\alpha_m} \) is defined by \( \Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_m \). If the integers be partitioned into two classes \( E_0 \) and \( E_1 \) according to whether \( \Omega(n) \equiv 0, 1 \mod 2 \), and \( E_0(x), E_1(x) \) be the corresponding counting functions, it follows that \( E_i(x) = (x/2) + \text{error} \). The error is \( O(x \exp \left( -a(\log x)^{1/2} \right) ) \) certainly, and on the Riemann hypothesis is \( O(x^{1/2+\epsilon}) \). This becomes evident when one considers \( \zeta(2s)/\zeta(s) \), which is the generating function for \( E_0(x) - E_1(x) \).

However, there is no "analogy" on the "error term" if the partitioning follow the residues of a number larger than 2, as we shall show. In fact, we shall establish the following

**Theorem.** If for any \( q \geq 3 \) we partition the integers into \( q \) classes \( \{G_{q,i}\} \), \( (i = 0, 1, \cdots, q-1) \), according to whether \( \Phi(n) = 0, 1, \cdots, q-1 \mod q \) and let \( C_{q,i}(x) \) be the corresponding counting functions, it follows that

\[
C_{q,i}(x) - x/q = \Omega_{q}(x/\log^r x),
\]

\( (i = 0, 1, \cdots, q-1) \), where \( r = 1 - \cos(2\pi/q) \).

The leading term \( x/q \), with error of \( o(x) \), has already been established by several investigators \([1;2]\) who made use only of elementary (non "complex-variable") arguments.

Specifically, we shall here actually compute the remainder term for \( q = 3 \). For larger values the computation is more complicated only with respect to notation, and we shall merely state the result.

Write \( \omega_1 = \exp(2\pi i/3) = (-1/2) + i(3^{1/2}/2) \), and \( \omega_2 = \omega_1^2 \). Define

\[
F(s) = \prod_p \left(1 - \frac{\omega_1}{p^s}\right)^{-1}
\]

so that

\[
C_{3,0}(x) + \omega_1 C_{3,1}(x) + \omega_2 C_{3,2}(x) = \frac{1}{2\pi i} \int_{2-\epsilon i}^{2+\epsilon i} F(s) \frac{x^s}{s} \, ds,
\]

where of course Cauchy mean value is understood. We shall show that \( F(s) \) has an algebraic singularity at \( s = 1 \), and behaves somewhat like \( (s-1)^{-\omega_1} \) in the neighborhood of that point. To this effect, expand
the logarithm of \( F(s) \) and group the terms according to the coefficients \( \omega_1, \omega_2, 1 \). Then from

\[
\sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns)
\]

it will follow that \( \log F(s) = \omega_1 \log \zeta(s) + \log g(s) \) where \( \log g(s) \) is regular and bounded in absolute value for \( \sigma \geq 1/2 + \epsilon \). So \( F(s) = (\zeta(s))^{\omega_1} g(s) / s = (s - 1)^{-\omega_1} h(s) \) and note that \( h(s) \) is regular and bounded in absolute value in the neighborhood of \( s = 1 \). Further, \( h(s) \) does not vanish in this region. Thus

\[
h(s) = k_0 + k_1(1 - s) + \cdots + k_n(1 - s)^n + \cdots \quad (k_0 \neq 0),
\]
valid for \( |1 - s| \leq 1 - a \).

Our problem is thus reduced to something similar to that of estimating the number of integers which are the sum of two squares \([3]\). Run the contour of integration from \( 2 - i \infty \) to \( 1 - a \geq 1/2 + \epsilon \), following the path \( \sigma = 1 - (a / \log(e + |s|)) \). From \( 1 - a \), run to \( 1 - \eta \), then on the circle of radius \( \eta \) about the point \( s = 1 \), then from \( 1 - \eta \), back to \( 1 - a \), and from there to \( 2 + i \infty \). Now this integral (except for the parts running along the real axis and about \( s = 1 \)) is \( O(x \exp[-a(\log x)^{1/2}]) \) certainly, and on the Riemann hypothesis is \( O(x^{1/2 + \epsilon}) \). For the rest, consider the first term of our integral

\[
\frac{k_0}{2\pi i} \int_{1-a}^{1-\eta} (s - 1)^{-\omega_1} x^s ds.
\]

Factor out \( (s - 1)^{-\omega_1} \) from \( (s - 1)^{-\omega_1} \) and make the substitution \( t = (1 - s) \log x \). For \( \eta = 0 \) this is equal to

\[
\frac{k_0'}{(1 - s)^{\omega_1}} \frac{x}{(\log x)^{1 - \omega_1}} + O(x^{1-a+\epsilon})
\]

\[
= k_0'' \frac{x}{(\log x)^{3/2}} \exp \left( i \frac{3^{1/2}}{2} \log \log x \right) + O(x^{1-a+\epsilon}),
\]
where \( k_0', k_0'' \neq 0 \).

The integral taken about the point \( s = 1 \) goes to zero with \( \eta \), but upon returning to the point \( 1 - \eta \) a new factor \( \neq 1 \) now multiplies the integrand. Thus the integral returning along the real axis merely changes the constant \( k_0'' \) to, say, \( m_0 \neq 0 \).

Repeating for the other terms in \( (1 - s)^n \) in the series we obtain the asymptotic expansion.
\[
\frac{x}{(\log x)^{3/2}} \left\{ \exp \left( i \frac{3^{1/2}}{2} \log \log x \right) \right\} \left( m_0 + \frac{m_1}{\log x} + \cdots + \frac{m_n}{\log^n x} + \cdots \right).
\]

To isolate the function \( C_{3,0}(x) \), for example, write

\[
G(s) = \prod_p \left( 1 - \frac{\omega_p}{p^s} \right)^{-1}.
\]

This will yield an asymptotic expansion identical with the above, except for a replacement of the constants \( m_j \) by \( \bar{m}_j \) (their complex conjugates) and a replacement of \( i \) by \( -i \) in the exponent of \( e \).

The function \( G(s) \) is the generating function for \( C_{3,0}(x) + \omega_2 C_{3,1}(x) + \omega_3 C_{3,2}(x) \). Thus \( F(s) + G(s) \) is the generating function for \( 3C_{3,0}(x) - [x] \). Thus

\[
3C_{3,0}(x) - x \sim \frac{x}{(\log x)^{3/2}}
\]

\[
\cdot \left\{ \left( \cos \left( \frac{3^{1/2}}{2} \log \log x \right) \right) \left( a_0 + \frac{a_1}{\log x} + \cdots + \frac{a_n}{\log^n x} + \cdots \right)
\]

\[
- \left( \sin \left( \frac{3^{1/2}}{2} \log \log x \right) \right) \left( b_0 + \frac{b_1}{\log x} + \cdots + \frac{b_n}{\log^n x} + \cdots \right) \}
\]

where \( a_j + ib_j = 2m_j \neq 0 \).

For \( q > 3 \) the analogous computations yield

\[
qC_{q,0}(x) - x \sim \frac{x}{(\log x)^{q/2}}
\]

\[
\cdot \sum_{j=1}^{[(q-1)/2]} \left\{ \left( \cos (v_j \log \log x) \right) \left( a_{0j} + \frac{a_{1j}}{\log x} + \cdots \right)
\]

\[
- \left( \sin (v_j \log \log x) \right) \left( b_{0j} + \frac{b_{1j}}{\log x} + \cdots \right) \}
\]

where \( r_j = 1 - \cos (2\pi j/q); v_j = \sin (2\pi j/q) \); and at least one of the numbers \( a_{0j}, b_{0j} \) is different from zero. This establishes our theorem.

For completeness, we mention the other measure of “compositeness” in use. If \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_m^{\alpha_m}, (\alpha_i > 0) \) we could also define \( \Omega^*(n) = m \), that is to say the number of distinct primes dividing \( n \), not counting multiplicity. It suffices to consider the generating functions

\[
H(s) = \prod_p \left( 1 + \frac{\omega_1}{p^s - 1} \right)
\]

and
\[ I(s) = \prod_p \left( 1 + \frac{\omega_2}{p^s - 1} \right) \]

which will yield a similar formula for \( q = 3 \). The general case likewise holds.

Finally we mention the "square-free" case. The square-frees are counted by the function \( Q(x) \). Partition them into three classes \( Q_{3,0}, Q_{3,1}, Q_{3,2}, \) etc. Here it suffices to note that

\[ \prod_p \left( 1 + \frac{\omega_1}{p^s} \right) = \frac{F(s)}{G(2s)} \]

and

\[ \prod_p \left( 1 + \frac{\omega_2}{p^s} \right) = \frac{G(s)}{F(2s)} \]

where \( F \) and \( G \) are defined as before. This again leads to a similar formula, with \( x/3 \) replaced by \( Q(x)/3 \). The general case likewise holds.

**References**


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