A NOTE ON THE COMPOSITENESS OF NUMBERS

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The "compositeness" of the number $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ is defined by $\Omega(n)=\alpha_1+\alpha_2+\cdots+\alpha_m$. If the integers be partitioned into two classes E_0 and E_1 according to whether $\Omega(n)\equiv 0, 1 \pmod 2$, and $E_0(x)$, $E_1(x)$ be the corresponding counting functions, it follows that $E_i(x)=(x/2)+\text{error}$. The error is $O(x \exp[-a(\log x)^{1/2}])$ certainly, and on the Riemann hypothesis is $O(x^{1/2+\epsilon})$. This becomes evident when one considers $\zeta(2s)/\zeta(s)$, which is the generating function for $E_0(x)-E_1(x)$.

However, there is no "analogy" on the "error term" if the partitioning follow the residues of a number larger than 2, as we shall show. In fact, we shall establish the following

THEOREM. *If for any $q\geqq 3$ we partition the integers into q classes $\{C_{q,i}\}$, $(i=0, 1, \cdots, q-1)$, according to whether $\Omega(n)\equiv 0, 1, \cdots, q-1 \pmod q$ and let $C_{q,i}(x)$ be the corresponding counting functions, it follows that*

$$C_{q,i}(x) - x/q = \Omega_{\pm}(x/\log^r x),$$

$(i=0, 1, \cdots, q-1)$, *where* $r=1-\cos(2\pi/q)$.

The leading term x/q, with error of $o(x)$, has already been established by several investigators [1; 2] who made use only of elementary (non "complex-variable") arguments.

Specifically, we shall here actually compute the remainder term for $q=3$. For larger values the computation is more complicated only with respect to notation, and we shall merely state the result.

Write $\omega_1=\exp(2\pi i/3)=(-1/2)+i(3^{1/2}/2)$, and $\omega_2=\omega_1^2$. Define

$$F(s) = \prod_p \left(1-\frac{\omega_1}{p^s}\right)^{-1}$$

so that

$$C_{3,0}(x)+\omega_1 C_{3,1}(x)+\omega_2 C_{3,2}(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s)\frac{x^s}{s}\,ds,$$

where of course Cauchy mean value is understood. We shall show that $F(s)$ has an algebraic singularity at $s=1$, and behaves somewhat like $(s-1)^{-\omega_1}$ in the neighborhood of that point. To this effect, expand

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the logarithm of $F(s)$ and group the terms according to the coefficients ω_1, ω_2, 1. Then from

$$\sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns)$$

it will follow that $\log F(s) = \omega_1 \log \zeta(s) + \log g(s)$ where $\log g(s)$ is regular and bounded in absolute value for $\sigma \geqq 1/2+\epsilon$. So $F(s) = (\zeta(s))^{\omega_1} g(s)$; $g(s) \neq 0$ for $\sigma \geqq 1/2+\epsilon$. Write $f(s) = F(s)/s = (s-1)^{-\omega_1} h(s)$ and note that $h(s)$ is regular and bounded in absolute value in the neighborhood of $s=1$. Further, $h(s)$ does not vanish in this region. Thus

$$h(s) = k_0 + k_1(1-s) + \cdots + k_n(1-s)^n + \cdots \qquad (k_0 \neq 0),$$

valid for $|1-s| \leqq 1-a$.

Our problem is thus reduced to something similar to that of estimating the number of integers which are the sum of two squares [**3**]. Run the contour of integration from $2-i\infty$ to $1-a \geqq 1/2+\epsilon$, following the path $\sigma = 1-(a/\log(e+|t|))$. From $1-a$, run to $1-\eta$, then on the circle of radius η about the point $s=1$, then from $1-\eta$, back to $1-a$, and from there to $2+i\infty$. Now this integral (except for the parts running along the real axis and about $s=1$) is $O(x \exp[-a(\log x)^{1/2}])$ certainly, and on the Riemann hypothesis is $O(x^{1/2+\epsilon})$. For the rest, consider the first term of our integral

$$\frac{k_0}{2\pi i} \int_{1-a}^{1-\eta} (s-1)^{-\omega_1} x^s ds.$$

Factor out $(-1)^{-\omega_1}$ from $(s-1)^{-\omega_1}$ and make the substitution $t=(1-s)\log x$. For $\eta=0$ this is equal to

$$\frac{k_0'(-1)^{-\omega_1}}{2\pi i} \cdot \frac{x}{(\log x)^{1-\omega_1}} + O(x^{1-a+\epsilon})$$
$$= k_0'' \frac{x}{(\log x)^{3/2}} \exp\left(i \frac{3^{1/2}}{2} \log\log x\right) + O(x^{1-a+\epsilon}),$$

where k_0', $k_0'' \neq 0$.

The integral taken about the point $s=1$ goes to zero with η, but upon returning to the point $1-\eta$ a new factor $\neq 1$ now multiplies the integrand. Thus the integral returning along the real axis merely changes the constant k_0'' to, say, $m_0 \neq 0$.

Repeating for the other terms in $(1-s)^n$ in the series we obtain the asymptotic expansion

$$\frac{x}{(\log x)^{3/2}}\left\{\exp\left(i\frac{3^{1/2}}{2}\log\log x\right)\right\}\left(m_0+\frac{m_1}{\log x}+\cdots+\frac{m_n}{\log^n x}+\cdots\right).$$

To isolate the function $C_{3,0}(x)$, for example, write

$$G(s) = \prod_p \left(1 - \frac{\omega_2}{p^s}\right)^{-1}.$$

This will yield an asymptotic expansion identical with the above, except for a replacement of the constants m_j by $\bar{m}_j$ (their complex conjugates) and a replacement of i by $-i$ in the exponent of e.

The function $G(s)$ is the generating function for $C_{3,0}(x)+\omega_2 C_{3,1}(x)+\omega_1 C_{3,2}(x)$. Thus $F(s)+G(s)$ is the generating function for $3C_{3,0}(x)-[x]$. Thus

$$3C_{3,0}(x) - x \sim \frac{x}{(\log x)^{3/2}}$$
$$\cdot\left\{\left(\cos\left(\frac{3^{1/2}}{2}\log\log x\right)\right)\left(a_0+\frac{a_1}{\log x}+\cdots+\frac{a_n}{\log^n x}+\cdots\right)\right.$$
$$\left.-\left(\sin\left(\frac{3^{1/2}}{2}\log\log x\right)\right)\left(b_0+\frac{b_1}{\log x}+\cdots+\frac{b_n}{\log^n x}+\cdots\right)\right\}$$

where $a_j+ib_j=2m_j\neq 0$.

For $q>3$ the analogous computations yield

$$qC_{q,0}(x) - x$$
$$\sim \sum_{j=1}^{[(q-1)/2]} \frac{x}{(\log x)^{r_j}}\left\{(\cos(v_j\log\log x))\left(a_{0j}+\frac{a_{1j}}{\log x}+\cdots\right)\right.$$
$$\left.-(\sin(v_j\log\log x))\left(b_{0j}+\frac{b_{1j}}{\log x}+\cdots\right)\right\}$$

where $r_j=1-\cos(2\pi j/q)$; $v_j=\sin(2\pi j/q)$; and at least one of the numbers a_{01}, b_{01} is different from zero. This establishes our theorem.

For completeness, we mention the other measure of "compositeness" in use. If $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, $(\alpha_i>0)$ we could also define $\Omega^*(n)=m$, that is to say the number of *distinct* primes dividing n, not counting multiplicity. It suffices to consider the generating functions

$$H(s) = \prod_p \left(1+\frac{\omega_1}{p^s-1}\right)$$

and

$$I(s) = \prod_p \left(1 + \frac{\omega_2}{p^s - 1}\right)$$

which will yield a similar formula for $q=3$. The general case likewise holds.

Finally we mention the "square-free" case. The square-frees are counted by the function $Q(x)$. Partition them into three classes $Q_{3,0}$, $Q_{3,1}$, $Q_{3,2}$, etc. Here it suffices to note that

$$\prod_p \left(1 + \frac{\omega_1}{p^s}\right) = \frac{F(s)}{G(2s)}$$

and

$$\prod_p \left(1 + \frac{\omega_2}{p^s}\right) = \frac{G(s)}{F(2s)}$$

where F and G are defined as before. This again leads to a similar formula, with $x/3$ replaced by $Q(x)/3$. The general case likewise holds.

References

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3. E. Landau, *Primzahlen*, Chelsea reprint, p. 645 ff.

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