ON A GENERALIZATION OF THE NOTION OF H^*-ALGEBRA[1]

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1. **Introduction.** An H^*-algebra of W. Ambrose [1] has the property that the orthogonal complement of an ideal is an ideal of the same kind. The present work is an attempt to characterize an H^*-algebra in terms of this property. For this purpose it is necessary to generalize the concept of H^*-algebra by introducing so-called two-sided H^*-algebras. We assume merely that there are two involutions in the algebra: right $x \to x^r$ and left $x \to x^l$ such that $(yx, z) = (y, zx^r)$ and $(xy, z) = (y, x^l z)$. It is possible to characterize two-sided H^*-algebra in terms of the above relation imposed on ideals by making some additional assumptions on the ideal annihilators. Since every simple two-sided H^*-algebra is an H^*-algebra with the same topology we have also found a new characterization of a proper H^*-algebra.

The notation is adopted essentially from [1; 6] and [3] but unlike Ambrose we do not require that ideals be closed. We shall make a distinction between a minimal ideal and a minimal closed ideal. Also it is understood that a proper ideal is not dense in whole algebra and that an idempotent is a nonzero element.

2. **Complemented and right complemented algebra. First structure theorem.**

DEFINITION 1. Let A be a Banach algebra which is a Hilbert space. We shall call A a *right complemented algebra* (*r. c. algebra*) if it has the property that the orthogonal complement of every right ideal is again a right ideal. Similarly we define a *left complemented algebra* (*l. c. algebra*). We shall call an algebra *complemented* if it is at the same time r. c. and l. c.

As an example of a right complemented algebra one can take a right H^*-algebra introduced by M. F. Smiley [6].

EXAMPLE 1. Let α be a positive norm-increasing bounded operator on a Hilbert space H and let A be the algebra of operators of the Hilbert Schmidt type on H. Then A is a right H^*-algebra (hence a r. c. algebra) in the scalar product $(a, b) = [\alpha a, b]$, where [,] denote a trace scalar product of the Hilbert Schmidt operators: $[a, b] = \text{tr}\,(a^* b)$.

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[1] Most of the results in this paper were taken from the author's doctoral thesis, Harvard University, 1954, written under the direction of Professor L. H. Loomis.

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DEFINITION 2. A right complemented algebra A will be called *proper* if $r(A) = \{x \in A \mid Ax = (0)\} = (0)$.

LEMMA 1. *The orthogonal complement I^p of a two-sided ideal I in a proper right complemented algebra A is a two-sided ideal and is identical with both the left and the right annihilator of I.*

PROOF. We know already that I^p is a right ideal. Since $I^p I \subset I^p \cap I = (0)$, I^p is contained in $l(I)$ which is a two-sided ideal. Hence it is sufficient to prove that $I^p = l(I)$. Consider $I_1 = I \cap l(I)$, which is also a two-sided ideal. It is readily verified that $I_1 \subset r(A) = (0)$. Thus $l(I) = I^p$. It follows at once that $I^p = r(I)$.

It is easy to see that a semi-simple r. c. algebra is proper and that every r. c. algebra is a direct sum of its radical and a semi-simple r. c. algebra. Also it is always true for a proper r. c. algebra A that $l(A) = (0)$.

We shall say that an element $x \in A$ is left self-adjoint if $(xy, z) = (y, xz)$ holds for every $y, z \in A$. An element e will be called a left projection if it is idempotent and left self-adjoint.

LEMMA 2. *Let R be a proper closed regular right ideal in an r. c. algebra A and let e be its relative identity such that $e \in R^p$. Then e is a left projection.*

PROOF. Since e is a relative identity we have $ex - x \in R$ for every $x \in A$. It follows that $eR = 0$ and $eR^p = R^p$; if $x \in R$, then $ex \in R$ and, since $ex \in R^p$ also, we have $ex = 0$; if $x \in R^p$, then $ex - x \in R^p \cap R$ and hence $ex - x = 0$, $ex = x$. In particular $e^2 = e$. Also e is left self-adjoint for if we consider $x, y \in A$ and write $x = x_1 + x_2$, $y = y_1 + y_2$ with $x_1, y_1 \in R^p$, $x_2, y_2 \in R$, we have:

$$(ex, y) = (ex_1 + ex_2, y_1 + y_2) = (x_1, y_1 + y_2) = (x_1, y_1).$$
$$(x, ey) = (x_1 + x_2, ey_1 + ey_2) = (x_1 + x_2, y_1) = (x_1, y_1).$$

LEMMA 3. *Every semi-simple r. c. algebra A contains a left projection.*

PROOF. Let x be an element in A which does not have a right quasi-inverse. Let R be the closure of the regular right ideal $\{xy + y \mid y \in A\}$. Then $-x$ is a relative identity of R. We write $-x = e + u$ with $e \in R^p$, $u \in R$; then one can easily check that e also is a relative identity of R. Hence e is a left projection.

Now we define the double orthogonality and the primitivity of a projection as in [1]. Also as in [1] we show that every left projection can be expressed as a finite sum of doubly orthogonal primitive left projections, and the closed right ideal eA, where e is a projection, is a

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minimal closed ideal if and only if e is primitive. From the first part of the last statement it follows that every semi-simple r. c. algebra contains a primitive left projection.

THEOREM 1. *Every semi-simple r. c. algebra A is a direct sum of simple r. c. algebras each of which is a two-sided ideal in A.*

PROOF. Let e be a primitive left projection in A and let I be the smallest closed two-sided ideal containing e. It is easy to see that I is a minimal closed two-sided ideal. Then I^p is also a semi-simple r. c. algebra. Using Zorn's lemma we complete the proof.

This is the first structure (Wedderburn) theorem for r. c. algebras.

3. **Two-sided H^*-algebras.**

DEFINITION 3. A Banach algebra A is called a *two-sided* H^*-algebra if A is a Hilbert space and if for every $a \in A$ there are elements a^l and a^r in A such that $(ab, c) = (b, a^l c)$ and $(ba, c) = (b, ca^r)$ hold for every $b, c \in A$.

THEOREM 2. *Every proper right H^*-algebra A is a two-sided H^*-algebra.*

PROOF. Let $x \in A$ and let N_1 be the linear space spanned by x. Let $M_1 = N_1^p$, $M_2 = M_1^r = \{y \in A \mid y^r \in M_1\}$ and $N_2 = M_2^p$. Then N_2 is one-dimensional. Take $u \in N_2$ so that $(x, x) = (x^r, u)$ (note that $(x^r, u) = 0$ for all $u \in N_2$ is impossible). We shall show that $u = x^l$. Let $y, z \in A$; then $zy^r = \lambda x + v$ where λ is some complex number and $v \in M_1$. It follows that $yz^r = \bar{\lambda} x^r + v^r$ with $v^r \in M_2$. Then we have: $(xy, z) = (x, zy^r)$ $= (x, \lambda x + v) = (x, \lambda x) = \bar{\lambda}(x, x) = \bar{\lambda}(x^r, u) = (\bar{\lambda} x^r, u) = (\bar{\lambda} x^r + v^r, u)$ $= (yz^r, u) = (y, uz)$.

4. **Well-complemented algebra. Second structure theorem.** It turns out that in order to prove the second structure theorem for (right, left) complemented algebras it is necessary to introduce a new axiom. The algebras with the new axiom will be called well-complemented.

DEFINITION 4. A semi-simple r. c. algebra A will be called *right well-complemented* (*r. w. c.*) if every proper right (left) ideal in A has a nonzero left (right) annihilator. A c. algebra will be called *well-complemented* (*w. c.*) if it is r. w. c.

A two-sided H^*-algebra furnishes an example of a w. c. algebra.

Now we prove a series of lemmas which will be used to prove the second structure theorem.

LEMMA 4. *Let L be a left ideal in a Banach algebra A such that every*

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member of L has a right quasi-inverse. Then L is contained in the radical of A.

PROOF. The lemma follows from the fact that if xy has a right quasi-inverse u then yx has also a right quasi-inverse $v=-(yx+yux)$.

LEMMA 5. *Every closed nonzero right ideal R in an r. w. c. algebra A contains a left projection.*

PROOF. Since A is semi-simple $l(R^p)$ contains an element x which does not have a right quasi-inverse. Consider the closed regular right ideal R_1 = closure of $\{xy+y \mid y\in A\}$ of which $-x$ is a relative identity. Since $xR^p=(0)$, we have $R^p\subset R_1$ and hence $R_1^p\subset R$. We write $-x=e+u$ with $e\in R_1^p$, $u\in R_1$; then e is again a relative identity of R_1. By Lemma 2 e is a left projection.

LEMMA 6. *If e is a primitive idempotent in an r. w. c. algebra A, then the right ideal $P=eA$ is a minimal right ideal.*

PROOF. We know already that P is a minimal *closed* ideal. So it is sufficient to show that if R is an ideal dense in P then $e\in R$. In this case we can find $x\in R$ such that $x-e$ has a right quasi-inverse y. Then $xy+x-e=0$ and hence $e\in R$.

Combining Lemma 6 with the technique used in [1] we prove:

LEMMA 7. *Let $\{e_i\}$ be a family of primitive idempotents in a simple algebra A. Let $A_{ij}=e_iAe_j$. Then:*
(i) *Each A_{ii} is isomorphic to the complex field;*
(ii) *Each A_{ij} is one-dimensional;*
(iii) $A_{ij}A_{jk}=A_{ik}$.

We shall say that an element x in a c. algebra A has a left adjoint if there exists an element $x^l\in A$ such that $(xy, a)=(y, x^lz)$ holds for all $y, z\in A$.

LEMMA 8. *If e is a primitive left projection in an r. w. c. algebra A then every element in Ae has a left adjoint.*

PROOF. Let $x\in Ae$. Consider the right ideal $R=xA=xeA$. We may assume that $ex\neq 0$ (otherwise we replace x by $y=x+e$ and prove that y has a left adjoint). Then since $exe=\lambda e$ we have $eR=eA$ from which it follows that R is closed. Hence R contains a left projection $f=xz=xez$, where z is some element in A. Then $fe\neq 0$ (otherwise $0=ef=exez=\lambda ez$, since ef is left adjoint of fe) and hence $fe=xeze=\mu x$ or $x=1/\mu fe$ from which follows that x has a left adjoint $x^l=1/\bar{\mu}ef$.

Now we are in position to prove the second structure theorem.

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THEOREM 3. *Every r. w. c. algebra is a two-sided H^*-algebra. In particular a simple r. w. c. algebra A is isomorphic to the algebra described in Example* 1.

PROOF. We follow the technique used in [1] and [6]. Let $\{e_i\}$ be a maximal family of doubly orthogonal primitive left projections in A. Consider $R=\sum_{i\in J} e_iA$, where J is the set of indices in $\{e_i\}$. Then R is closed. If $R\neq A$ then R^p contains a primitive left projection e. Then $(e_ie, e_ie)=(e, e_ie)=0$, $e_ie=0$ for every $i\in J$, i.e., e is doubly orthogonal to every e_i, which leads to a contradiction. Thus $R=A$. Consider $L=\sum_{i\in J} Ae_i$ and suppose $\bar{L}\neq A$. Then the right annihilator $r(L)$ is a nonzero right ideal. This simply means that there is an element $x\in A$ such that $e_ix=0$ for all e_i. Then for any $y\in e_iA$ we have $(e_iy, x)=(y, e_ix)=0$, i.e., x is orthogonal to all e_iA, hence to whole A, which is a contradiction. Thus L is dense in A. Let us use the notation $A_{ij}=e_iAe_j$; then $L=\sum_{i,j} A_{ij}$. It follows from Lemma 8 that every element in A_{ij} has a left adjoint (in A_{ji}). So we choose the matrix units $e_{ij}\in A_{ij}$ such that $e_{ii}=e_i$ and $e_{ij}^l=e_{ji}$. We define the matrix (α_{ij}) by setting $\alpha_{ij}=(e_{ki}, e_{kj})$. It is easy to see that α_{ij} does not depend upon k and that the matrix (α_{ij}) is self-adjoint. Any two elements in L have the form $x=\sum_{i,j} x_{ij}e_{ij}$ and $y=\sum_{i,j}y_{ij}e_{ij}$, where x_{ij} and y_{ij} are suitable complex numbers, and the scalar product has the form:

$$(x, y) = \sum x_{ik}\alpha_{kj}\bar{y}_{ij} = \operatorname{tr}(\mathbf{x}\boldsymbol{\alpha}\mathbf{y}^*),$$

where $\mathbf{x}$, $\mathbf{y}$ and $\boldsymbol{\alpha}$ here stand for matrices (x_{ij}), (y_{ij}) and (α_{ij}) respectively.

Now we shall show that (α_{ij}) represents a bounded operator on $L^2(J)$. For this purpose let us consider the conjugate-linear mapping T: $x\to x^l$ restricted to Ae_1, where 1 is some fixed index in J. Since every element in Ae_1 has a left adjoint T is defined everywhere on Ae_1; the range of T is a subset of e_1A (in fact one can show that the range is entire e_1A). The graph of T is closed: if $\langle x_n, x_n^l\rangle\to\langle x, u\rangle$, then $x_n\to x$ and $x_n^l\to u$, and for every $y, z\in A\,(x_ny, z)\to(xy, z)$ and $(y, x_n^lz)\to(y, uz)$ from which it follows that $x^l=u$, and so $\langle x, u\rangle$ also belongs to the graph of T. From the closed graph theorem it follows that T is continuous. Thus there exists a positive number M such that

$$(*)\qquad (x^l, x^l) \leqq M(x, x)$$

holds for all $x\in Ae_1$.

Now there is a natural 1-1 correspondence between elements of $L^2(J)$ and Ae_1, in which a member $x(i)$ of $L^2(J)$ corresponds to the element $x=\sum_{i\in J} x(i)e_{i1}$ and Ae_1. If $x(i)$ and $y(i)$ are finite sequences

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in $L^2(J)$ then $x^l = \sum_i \bar{x}(i)e_{1i}$, $y^l = \sum_i \bar{y}(i)e_{1i}$ and $(x^l, y^l) = \sum_{i,j} \bar{x}(i)\alpha_{ij}y(j)$. By the continuity of T $(x^l, y^l) = \sum_{i,j} \bar{x}(i)\alpha_{ij}y(j)$ holds for all $x(i), y(i)$ in $L^2(J)$. From (*) we have then $\sum_{i,j} x(i)\alpha_{ij}\bar{x}(j) \leqq \alpha_{11}M\sum_i |x(i)|^2$. Completing the proof as in [6] we show that L is an algebra of the type described in Example 1. Since L is complete we have $L=A$. So A is a left and hence a two-sided H^*-algebra.

To conclude this section we construct a complemented algebra which is not well-complemented.

EXAMPLE 2. Let $\tilde{\alpha}$ be some positive norm-increasing unbounded operator with domain dense in some Hilbert space H. Let A be the set of all operators a on H such that $a\tilde{\alpha}$ is an operator of the Hilbert Schmidt type. Then A is a complemented algebra in the scalar product $(a, b) = [a\tilde{\alpha}, b\tilde{\alpha}] = \mathrm{tr}(a\tilde{\alpha}(b\tilde{\alpha})^*)$. It is easy to see that there are two dense subsets of elements in A, every element of one having the left adjoint $a^l = a^*$ and every element of the other having the right adjoint $a^r = \tilde{\alpha}^2a^*\tilde{\alpha}^{-2}$ in A. It remains to show that A is not well-complemented. Let us denote by $\tilde{A}$ the algebra of all operators of the Hilbert Schmidt type on H, and let e be some left projection in A. It is easy to show that the Hilbert space H can be realized as $\tilde{A}e$ and so that if $\tilde{x} \in H$ corresponds to $x \in \tilde{A}e$ and α is any operator such that $\alpha(x)$ is defined then $\alpha(\tilde{x}) = \alpha x$. Since $\tilde{\alpha}$ is unbounded there exists an $a \in \tilde{A}e = Ae$ such that $\tilde{\alpha}a$ is not of the Hilbert Schmidt type. This means that a does not have the left adjoint. Then from Lemma 8 it follows that A is not well-complemented.

5. A special realization of a well-complemented algebra.

EXAMPLE 3. Let λ be a norm-decreasing linear transformation from a Hilbert space H_2 onto a Hilbert space H_1 which has a bounded inverse transformation μ from H_1 onto H_2. Let A be the set of all Hilbert Schmidt operators a from H_1 into H_2. Let us define the multiplication by $a \circ b = a\lambda b$. Then A is a w. c. algebra in the scalar product $(a, b) = \mathrm{tr}\,(ab^*)$. All the laws of an algebra are easily verified; $\|ab\| \leqq \|a\|\,\|b\|$ follows from the fact that λ is norm-decreasing. The right and the left adjoint of an element $a \in A$ are defined by $a^r = \mu a^*\lambda^*$ and $a^l = \lambda^*a^*\mu$.

It turns out that every simple w. c. algebra A is of the above form. This is shown in the next theorem.

THEOREM 4. *Every simple w. c. algebra A is of the form described in Example* 3.

PROOF. Let $\{e_j\}$ and $\{f_k\}$ be maximal families of doubly orthogonal primitive left and right projections in A respectively. Then $A = \sum_{j,k} e_jAf_k$. We choose e_{ij} and f_{kl} as in Theorem 3, such that

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$e_{jj}=e_j$, $e^l_{ij}=e_{ji}$, $f_{kk}=f_k$, $f^r_{kl}=f_{lk}$. Let $J=(j)$ and $K=(k)$ be the sets of indices in $\{e_j\}$ and $\{f_k\}$ respectively; let $1\in J$ and $2\in K$ be some fixed indices. Choose $w_{12}\in e_1Af_2$ such that $\|w_{12}\|=1$ and let $w_{jk}=e_{j1}w_{12}f_{2k}$ for each $\langle j, k\rangle\in J\times K$. Then $w_{jk}\in e_jAf_k$ and $\|w_{jk}\|=1$. Since every e_jAf_k is one-dimensional, w_{jk} constitute an orthonormal base for A. Thus every $x\in A$ has the form $x=\sum_{j,k} x(j, k)w_{jk}$ and so the scalar product is of the form $(x, y)=\sum_{j,k} x(j, k)\bar{y}(j, k)$.

Now consider $w_{1k}w_{j2}$; since $w_{1k}w_{j2}\in e_1Af_2$ we have $w_{1k}w_{j2}=\lambda_{kj}w_{12}$ for some complex λ_{kj}. Multiplying both sides of the last equality with e_{i1} on the left and with f_{2l} on the right we get $w_{ik}w_{jl}=\lambda_{kj}w_{il}$. Thus the multiplication has the form $xy=\sum_{i,k} x(i, k)w_{ik} \sum_{j,l} y(j, l)w_{jl} =\sum_{i,l}(\sum_{k,j} x(i, k)\lambda_{kj}y(j, l))w_il$.

It remains to show that the matrix λ_{kj} regarded as an operator from $H_2=L^2(J)$ into $H_1=L^2(K)$ is norm-decreasing and has an inverse. Since $w_{ik}\in e_iA$ we have $w_{ik}=\sum_h \tau_{kh}e_{ih}$ for some matrix $\{\tau_{kh}\}$ which can be shown to be independent of i; also $e_{ih}=\sum_l \mu_{hl}w_{il}$ where $\{\mu_{hl}\}$ is independent of i. It is easy to show that $\{\mu_{hl}\}$ is the inverse of $\{\tau_{kh}\}$. From the other hand $\lambda_{kj}w_{il}=w_{ik}w_{jl}=\sum_h \tau_{kh}e_{ih}w_{jl}=\tau_{kj}e_{ij}w_{jl}=\tau_{kj}w_{il}$ which simply means that $\lambda_{kj}=\tau_{kj}$ and that $\{\mu_{hl}\}$ is an inverse of $\{\lambda_{kj}\}$. In order to show that $\{\lambda_{kj}\}$ is norm-decreasing we consider $\alpha_{ij}=(e_{1i}, e_{1j})=(\sum_k \mu_{ik}w_{1k}, \sum_l \mu_{jl}w_{1l})=\sum_k \mu_{ik}\bar{\mu}_{jk}$ which simply means that $\alpha=\mu\mu^*$. From the fact that α is positive and norm-increasing it is easy to derive that μ is also norm-increasing. The rest of the proof follows immediately.

Thus the algebra of Example 3 is essentially a most general w. c. algebra. All w. c. algebras are obtained by considering all possible direct sums of the algebras of the form of that described in Example 3.

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