CYLINDRIC AND POLYADIC ALGEBRAS

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1. **Introduction.** In recent years there have appeared two algebraizations of the first-order predicate calculus; i.e., the polyadic algebras of Halmos [**1**; **2**], and the cylindric algebras of Tarski [**3**; **4**]. While polyadic algebras are the algebraic version of the *pure* first-order calculus, cylindric algebras yield an algebraization of the first-order calculus *with equality*. Since the pure calculus does not contain any identifiable predicate, one cannot expect to find the algebraic analogue of an equality predicate in a general polyadic algebra. It is reasonable, however, to consider "adjoining" an equality predicate, in some sense, to a polyadic algebra, and ask if one then obtains a cylindric algebra. This is the procedure followed here. An e-algebra is defined as a polyadic algebra with an equality predicate. We show that every e-algebra is in a natural way a cylindric algebra. Conversely, it is shown that in the presence of an infinite supply of variables and a local finiteness condition, cylindric algebras are in a natural way e-algebras, and the correspondence obtained in this way between e-algebras and cylindric algebras is one-to-one.

2. **Polyadic algebras.** A *quantifier* (or, more explicitly, an *existential quantifier*) on a Boolean algebra A is a mapping $\exists: A \to A$ such that (1) $\exists 0 = 0$, (2) $p \leqq \exists p$, and (3) $\exists(p \wedge \exists q) = \exists p \wedge \exists q$ for all $p, q \in A$. A *polyadic algebra* is a quadruple $(A, I, S, \exists)$, where A is a Boolean algebra, I an arbitrary set whose elements are called *variables*, S is a mapping from transformations of I into itself to Boolean endomorphisms on A (the transformations need not be one-to-one nor onto), and $\exists$ is a mapping from subsets of I to quantifiers on A, satisfying the following conditions:

(P_1) $\exists(\varnothing)p = p$ for all $p \in A$ ($\varnothing$ shall denote the empty set throughout).

(P_2) $\exists(J \cup K) = \exists(J)\exists(K)$ for all subsets J and K of I.

(P_3) $S(\delta) = f$ (where δ is the identity transformation on I and f is the identity endomorphism on A).

(P_4) $S(\sigma)S(\tau) = S(\sigma\tau)$ for all transformations σ and τ on I.

(P_5) If $J \subset I$ and σ and τ are transformations on I which agree outside J, then $S(\sigma)\exists(J) = S(\tau)\exists(J)$.

(P_6) If $J \subset I$ and τ is a transformation which is one-to-one on $\tau^{-1}J$, then $\exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}J)$.

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If $p \in A$, then p will be said to be *supported by* the set J if $\exists(I-J)p = p$. We will say that p is *independent* of the set K if $\exists(K)p = p$, so that J supports p if and only if p is independent of $I-J$. A polyadic algebra will be called *locally finite* if each element p of the algebra is supported by some finite set J_p. A transformation τ will be called *finite* if τ agrees with δ outside some finite set. If i and j are elements of I, the transformation which maps i onto j and every other element of I (including j) onto itself will be called a *replacement* and denoted by (i/j). If I is infinite, the algebra will be said to have *infinite degree*. A quasi-polyadic algebra is a quadruple $(A, I, S, \exists)$, where A is a Boolean algebra, I a set, S a mapping from finite transformations on I to Boolean endomorphisms on A, and $\exists$ is a mapping from finite subsets of I to quantifiers on A, satisfying the conditions:

(Q_1) $\exists(\emptyset)p = p$ whenever $p \in A$.

(Q_2) $\exists(J \cup K) = \exists(J)\exists(K)$ whenever J and K are finite subsets of I.

(Q_3) $S(\delta) = f$.

(Q_4) $S(\sigma)S(\tau) = S(\sigma\tau)$ whenever σ and τ are finite transformations on I.

(Q_5) If σ and τ are finite transformations on I, if J is a finite subset of I, and $\sigma = \tau$ outside J, then $S(\sigma)\exists(J) = S(\tau)\exists(J)$.

(Q_6) If τ is a finite transformation on I, if J is a finite subset of I, and if τ is one-to-one on $\tau^{-1}J$, then $\exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}J)$.

(Q_7) If $p \in A$, then there exists a cofinite set J (i.e., $I-J$ is a finite set) such that $\exists(K)p = p$ whenever K is a finite subset of J.

We shall need the following result concerning quasi-polyadic algebras from [**2**].

THEOREM. *If* $(A, I, S, \exists)$ *is a quasi-polyadic algebra, then* (i) *there exists a mapping* S^* *from transformations on* I *to Boolean endomorphisms of* A *such that* $S^*(\tau) = S(\tau)$ *whenever* τ *is a finite transformation,* (ii) *there exists a mapping* $\exists^*$ *from subsets of* I *to quantifiers on* A *such that* $\exists^*(J) = \exists(J)$ *whenever* J *is a finite set,* (iii) *the quadruple* $(A, I, S^*, \exists^*)$ *is a locally finite polyadic algebra, and* (iv) *the mappings* S^* *and* $\exists^*$ *are uniquely determined by* (i), (ii), *and* (iii).

We shall also need the fact, established in [**2**], that if τ is a finite transformation on I and J is a finite subset of I, then there is a finite ordered collection $\{\tau_1, \cdots, \tau_n\}$ of replacements on I such that $\tau = \tau_1 \cdots \tau_n$ on J.

3. ***e*-Algebras.** If $e(\ ,\ )$ is the equality predicate for the first-order functional calculus with equality, then it is well known that $e(\ ,\ )$ is characterized by the reflexive and substitution properties. Moreover,

if we have $e(x, y)$, then the transformation which maps x onto z yields the equality of z and y; i.e., $e(z, y)$. More generally, the effect on $e(x, y)$ of a transformation on the variables is obtained by allowing the transformation to act on the variables x and y directly. These considerations furnish the motivation for Definitions 1 and 2. (Condition (2) of Definition 2 asserts essentially that if p is true and $i=j$, then p is true with i replaced by j; i.e., the substitution property.)

DEFINITION 1. Let $(A, I, S, \exists)$ be a polyadic algebra. A *binary predicate* for A is a function $p: I\times I\rightarrow A$ such that $S(\tau)p(i, j)=p(\tau i, \tau j)$ for every transformation τ on I.

DEFINITION 2. A polyadic algebra *with equality* (or, an *e-algebra*) is a polyadic algebra $(A, I, S, \exists)$ for which there exists a binary predicate e for A such that (1) $e(i, i)=1$ for all $i\in I$, and (2) $p\wedge e(i, j) \leqq S(i/j)p$ for all $i, j\in I$ and $p\in A$. We shall denote the e-algebra by $(A, I, S, \exists, e)$.

DEFINITION 3. A *cylindric algebra* is a Boolean algebra A, together with a function C from a set I to quantifiers on A, and a function $d: I\times I\rightarrow A$ such that (1) $C(h)C(j)=C(j)C(h)$, (2) $d(i, i)=1$, (3) $d(i, j) =C(k)[d(i, k)\wedge d(j, k)]$, and (4) $C(i)[p\wedge d(i, k)]\wedge C(i)[p'\wedge d(i, k)] =0$ whenever i, j, h, k are elements of I such that $i\neq k$ and $j\neq k$. The cylindric algebra will be denoted by (A, I, C, d).

DEFINITION 4. A cylindric algebra (A, I, C, d) will be called *locally finite* if for each $p\in A$, the set $\{j\in I \mid C(j)p=p\}$ is cofinite.

We note that Definitions 3 and 4 are in an obvious way equivalent to the definitions given by Tarski in [**3**].

DEFINITION 5. An e-algebra $(A, I, S, \exists, e)$ will be called *cylindrizable* if there exists a cylindric algebra (A_1, I_1, C, d) such that $A_1=A$, $I_1=I$, $d=e$, and $C(i)=\exists(i)$ for all $i\in I$.

DEFINITION 6. A cylindric algebra (A, I, C, d) will be called *equalizable* if there exists an e-algebra $(A_1, I_1, S, \exists, e)$ such that $A_1=A$, $I_1=I$, $e=d$, $S(i/j)p=C(i)[p\wedge d(i, j)]$ whenever $i\neq j$, and $\exists(i)=C(i)$ for all $i\in I$.

Let $(A, I, S, \exists, e)$ be an e-algebra. We shall need the following lemmas.

LEMMA 1. *Whenever $i\neq j$, $S(i/j)p=\exists(i)[p\wedge e(i, j)]$ for all $p\in A$.*

PROOF. $\exists(i)[p\wedge e(i, j)]\leqq \exists(i)S(i/j)p=S(i/j)p$, since $i\neq j$. Also, $S(i/j)p=S(i/j)p\wedge e(j, j)=S(i/j)[p\wedge e(i, j)]\leqq S(i/j)\exists(i)[p\wedge e(i, j)] =\exists(i)[p\wedge e(i, j)]$.

LEMMA 2. *For all $i, j\in I$, $e(i, j) =e(j, i)$.*

PROOF. By symmetry, it is sufficient to show that $e(i, j) \leqq e(j, i)$ for all $i, j \in I$. But $e'(j, i) \wedge e(i, j) \leqq S(i/j)e'(j, i) = e'(j, j) = 0$.

THEOREM 1. *Every e-algebra is cylindrizable.*

PROOF. Let $(A, I, S, \exists, e)$ be an e-algebra. We define $d = e$, and let $C(k) = \exists(k)$ for all $k \in I$. It is clear that C maps I into quantifiers on A which commute, and that $d(i, i) = e(i, i) = 1$ for all $i \in I$. If $i \neq k$, $j \neq k$, then, by Lemmas 1 and 2, $C(k)[d(i, k) \wedge d(j, k)] = \exists(k)[e(i, k) \wedge e(k, j)] = S(k/j)e(i, k) = e(i, j) = d(i, j)$. Finally, if $i \neq k$ and $p \in A$, we have $\exists(i)[p \wedge d(i, k)] \wedge \exists(i)[p' \wedge d(i, k)] = S(i/k)p \wedge S(i/k)p' = S(i/k)(p \wedge p') = 0$, so that (A, I, C, d) is a cylindric algebra.

4. **Cylindric algebras.** Let (A, I, C, d) be a locally finite cylindric algebra with I infinite. We shall show that (A, I, C, d) is equalizable. We let $e = d$, $\exists(\varnothing)p = p$, $\exists(j) = C(j)$, $S(j/j)p = p$, and $S(i/k)p = C(i)[p \wedge d(i, k)]$ whenever $p \in A$, and $i, j, k \in I$ such that $i \neq k$, and we define $S(\tau)p$ for $p \in A$ and τ a finite transformation on I, by finding a finite set of replacements on I, say $\{\tau_1, \cdots, \tau_n\}$ such that $\tau = \tau_1 \cdots \tau_n$ on some finite support of p and letting $S(\tau)p = S(\tau_1) \cdots S(\tau_n)p$. Such a finite set of replacements exists, as we have remarked above, but it will be necessary to show that the definition is unambiguous. If $J = \{j_1, \cdots, j_n\}$ is a finite subset of I, we define $\exists(J)$ by the equation $\exists(J) = C(j_1) \cdots C(j_n)$. Since the values of C commute, and since (as is easily verified) the product of two commuting quantifiers is again a quantifier, $\exists(J)$ is unambiguously defined and is a quantifier.

The proofs of the next four lemmas consist of straightforward computations, and are omitted.

LEMMA 3. *If $i \neq j$, then $S(i/j)$ is a Boolean endomorphism on A.*

LEMMA 4. (1) *Whenever $i \neq j$, $k \neq j$, $S(i/k)\exists(j) = \exists(j)S(i/k)$,* (2) *$S(j/i)\exists(j) = \exists(j)$ for all $i, j \in I$, and* (3) *$\exists(j)S(j/i) = S(j/i)$ whenever $i \neq j$.*

LEMMA 5. *If i, j, k, h are distinct elements of I, then* (1) *$S(i/j)S(k/h) = S(k/h)S(i/j)$,* (2) *$S(k/h)S(k/j) = S(k/j)$,* (3) *$S(k/j)S(k/j) = S(k/j)$,* (4) *$S(i/j)S(k/i) = S(k/j)S(i/j)$,* (5) *$S(i/j)S(k/j) = S(k/j)S(i/j)$.*

LEMMA 6. *If $\exists(j)p = p$, then $S(j/i)S(i/j)p = p$ whenever $p \in A$ and $i, j \in I$.*

DEFINITION 7. Let α be an ordered collection consisting of an even number of replacements on I, say $\alpha = \{\alpha_1, \cdots, \alpha_{2n}\}$, $n \geqq 0$, and J a finite subset of I. We shall say that α is *J-normal* if there are distinct

elements $k_1, \cdots, k_n \in J$, distinct elements $i_1, \cdots, i_n \in I-J$, and (not necessarily distinct) elements $j_1, \cdots, j_n \in J$ such that (1) $\alpha_r = (i_r/j_r)$, and (2) $\alpha_{n+r} = (k_r/i_r)$, $r = 1, 2, \cdots, n$.

DEFINITION 8. If $\alpha = \{\alpha_1, \cdots, \alpha_n\}$ and $\beta = \{\beta_1, \cdots, \beta_m\}$ are finite ordered collections of replacements on I, J any subset of I, and $p \in A$, we will say that α and β are *(p, J)-equivalent* if (1) $\alpha_1 \cdots \alpha_n j = \beta_1 \cdots \beta_m j$ whenever $j \in J$, and (2) $S(\alpha)p = S(\beta)p$, where $S(\alpha) = S(\alpha_1) \cdots S(\alpha_n)$ and $S(\beta) = S(\beta_1) \cdots S(\beta_m)$. If α and β are (p, I)-equivalent for every $p \in A$, we shall say that α and β are *equivalent*.

DEFINITION 9. If (i/j) is a replacement on I, we shall refer to i as the *essential domain* of (i/j), and to j as the *essential range* of (i/j).

Lemma 7 enables one to study the effect of a finite transformation τ on a finite set by examining the image of each element separately. The method is one commonly used in mathematical logic; i.e., mapping the element i first into another element j far from the scene of the action, and then mapping j into $\tau(i)$.

LEMMA 7. *Let α be a finite ordered collection of replacements, $p \in A$, and J a finite support of p which contains all essential domains and essential ranges of elements of α. Then there exists a finite ordered collection ϕ of replacements which is J-normal and (p, J)-equivalent to α. Moreover, if $\phi = \{\phi_1, \cdots, \phi_{2m}\}$, the essential domains of $\phi_1, \cdots, \phi_m$ may be chosen arbitrarily from $I-J$, provided they are distinct, and the essential domains of $\phi_{m+1}, \cdots, \phi_{2m}$ are all elements of J.*

PROOF. The proof consists of successively transforming α into various ordered collections, the last of which is ϕ, with the property that each is (p, J)-equivalent to the preceding one. The details are omitted.

Lemma 8 states essentially that if two transformations agree on a (finite) set P which supports an element q of A, except possibly on a subset K of P of which q is independent, then they produce the same effect on q.

LEMMA 8. *Let $\alpha = \{\alpha_1, \cdots, \alpha_{n_1}\}$ and $\alpha^* = \{\alpha_1^*, \cdots, \alpha_{m_1}^*\}$ be finite ordered collections of replacements on I, $p \in A$, P a finite support of p, and K any finite subset of I, such that $\alpha_1 \cdots \alpha_{n_1} j = \alpha_1^* \cdots \alpha_{m_1}^* j$ whenever $j \in P-K$. Then $S(\alpha)\,\exists(K)p = S(\alpha^*)\,\exists(K)p$ (cf. Definition 8).*

PROOF. Applying Lemma 7 to α and α^* and a finite set J which contains P and satisfies the hypotheses of Lemma 7 with respect to α and α^*, we obtain finite ordered collections β and β^*. These in turn can be transformed into collections γ and γ^* such that $\gamma = \gamma^*$. Since α, β, γ and α^*, β^*, γ^* are seen to be $(\exists(K)p, J)$-equivalent, it

follows that $S(\alpha)\exists(K)p = S(\beta)\exists(K)p = S(\gamma)\exists(K)p = S(\gamma^*)\exists(K)p = S(\beta^*)\exists(K)p = S(\alpha^*)\exists(K)p$.

COROLLARY 1. *The definition of $S(\tau)$ is unambiguous for every finite transformation τ.*

PROOF. Let $\alpha = \{\alpha_1, \cdots, \alpha_n\}$ and $\alpha^* = \{\alpha_1^*, \cdots, \alpha_m\}$ be finite ordered collections of replacements such that $\alpha_1 \cdots \alpha_n = \tau$ on the finite support Q of p, and $\alpha_1^* \cdots \alpha_m^* = \tau$ on the finite support Q^* of p. Then $P = Q \cap Q^*$ is a finite support of p, and $\alpha_1 \cdots \alpha_n = \tau = \alpha_1^* \cdots \alpha_m^*$ on P. We apply Lemma 8 with $K = \varnothing$, to obtain $S(\alpha)p = S(\alpha^*)p$.

COROLLARY 2. *If σ, τ are finite transformations on I which agree outside a finite set K, then $S(\sigma)\exists(K) = S(\tau)\exists(K)$.*

PROOF. Let $p \in A$. We find a finite ordered collection $\alpha = \{\alpha_1, \cdots, \alpha_n\}$ of replacements such that $\alpha_1 \cdots \alpha_n = \sigma$ on a finite support P_1 of p, and a finite ordered collection $\beta = \{\beta_1, \cdots, \beta_m\}$ such that $\beta_1 \cdots \beta_m = \tau$ on a finite support P_2 of p. Then $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ on $P - K$, where $P = P_1 \cap P_2$. It follows from Lemma 8 that $S(\sigma)\exists(K)p = S(\alpha)\exists(K)p = S(\beta)\exists(K)p = S(\tau)\exists(K)p$.

LEMMA 9. *Let τ be a finite transformation on I, J a finite subset of I. If τ is one-to-one on $\tau^{-1}J$, then $S(\tau)\exists(\tau^{-1}J) = \exists(J)S(\tau)$.*

PROOF. If $J = \varnothing$, the lemma is trivial. Assume first that $J = \{j\}$. Let $p \in A$, and let α be a finite ordered collection of replacements, $\alpha = \{\alpha_1, \cdots, \alpha_n\}$, such that (1) $\alpha_1 \cdots \alpha_n = \tau$ on a support K_1 of p, and (2) $(\alpha_1 \cdots \alpha_n)^{-1}j = \tau^{-1}j$. (It is possible to find such a collection, for example, by considering $\tau | (I - J)$.) Let K be a finite support of p which includes all essential domains and essential ranges of elements of α, as well as j and $k = \tau^{-1}j = (\alpha_1 \cdots \alpha_n)^{-1}j$. By Lemma 7, we can find a finite ordered collection of replacements $\beta = \{\beta_1, \cdots, \beta_{2m}\}$ which is K-normal and $(\exists(j)p, K)$-equivalent to α. The proof for $J = \{j\}$ then follows from Lemmas 4, 5, and 6 by consideration of the cases $j = k$ and $j \neq k$.

If $J = \{j_1, \cdots, j_n\}$, $n \geqq 1$, and the lemma holds for all sets J_1 with fewer than n elements, let τ be one-to-one on $\tau^{-1}J$. Then τ is one-to-one on $\tau^{-1}j_1$ and on $\tau^{-1}J_1$, where $J_1 = J - \{j_1\}$, and $\exists(J)S(\tau) = \exists(J_1)\exists(j_1)S(\tau) = \exists(J_1)S(\tau)\exists(\tau^{-1}j_1) = S(\tau)\exists(\tau^{-1}J_1 \cup \tau^{-1}j_1) = S(\tau)\exists(\tau^{-1}J)$.

We are now in a position to prove the principal theorem.

THEOREM 2. *Every locally finite cylindrical algebra of infinite degree is equalizable.*

PROOF. Let (A, I, C, d) be a locally finite cylindric algebra with I infinite. We define e, S, and $\exists(j)$ for $j \in I$ as in Definition 6, and for J finite, $J = \{j_1, \cdots, j_n\}$, we define $\exists(J) = C(j_1) \cdots C(j_n)$. It follows from our earlier remarks and Corollary 1 to Lemma 8 that the definitions of $\exists(J)$ and $S(\tau)$ are unambiguous and that $\exists(J)$ is a quantifier for any finite subset J of I and any finite transformation τ on I. An easy induction based on Lemma 3 shows that $S(\tau)$ is a Boolean endomorphism for any finite τ. We shall see that the postulates for a quasi-polyadic algebra are satisfied by $(A, I, S, \exists)$, and it will follow from the theorem on quasi-polyadic algebras quoted above that $(A, I, S, \exists)$ determines a unique polyadic algebra.

Since $S(j/j)p = p$ for all $j \in I$ and $p \in A$, it follows that $S(\delta)p = p$, so that Q_1 holds. Postulates Q_2, Q_3, Q_4, and Q_7 follow immediately from the definitions, and Q_5 is Corollary 2 to Lemma 8, while Q_6 is Lemma 9. We must show that e is a binary predicate satisfying conditions (1) and (2) of Definition 2. Since $e = d$, we know that e maps $I \times I$ into A, and $e(i, i) = 1$ for all $i \in I$. To show that $S(\tau)e(i, j) = e(\tau i, \tau j)$, it follows from the definition of S that it is sufficient to verify the equation $S(k/h)e(i, j) = e((k/h)i, (k/h)j)$ for all $i, j, h, k \in I$. If $k \notin \{i, j\}$ or if $k = h$, then the equation holds trivially, since $e(i, j)$ is supported by $\{i, j\}$. Suppose, then, that $k = i$, $k \neq h$. Then $S(k/h)e(i, j) = \exists(i)[d(i, j) \wedge d(i, h)] = \exists(i)[d(j, i) \wedge d(h, i)] = d(j, h) = d(h, j) = e((k/h)i, (k/h)j)$. The case $k = j$, $k \neq h$ is similar. Now suppose $p \in A$. Then $p \wedge e(i, j) \leqq \exists(i)[p \wedge d(i, j)] = S(i/j)p$ whenever $i \neq j$, and the inequality holds trivially when $i = j$.

THEOREM 3. *Let $\mathfrak{A} = (A, I, S, \exists, e)$ be a locally finite e-algebra of infinite degree. Let $\mathfrak{B} = (A, I, C, d)$ be the locally finite cylindric algebra of infinite degree arising from $\mathfrak{A}$ by means of Definition* 5 (*cf. Theorem* 1). *Let $\mathfrak{A}_1$ be the e-algebra arising from $\mathfrak{B}$ by means of Definition* 6 (*cf. Theorem* 2). *Then $\mathfrak{A}_1 = \mathfrak{A}$.*

PROOF. Let $\mathfrak{A}_1 = (A, I, \exists_1, S_1, e_1)$. It follows from definitions that $\exists_1(k) = C(k) = \exists(k)$ for all $k \in I$, and therefore that $\exists_1(J) = \exists(J)$ for any finite J. Also, we have $e_1 = d = e$. If $p \in A$ and $i, j \in I$, $i \neq j$, then $S_1(i/j)p = C(i)[p \wedge d(i, j)] = \exists(i)[p \wedge e(i, j)] = S(i/j)p$ by the definition of S_1 and Lemma 1. An easy induction shows that $S_1(\tau) = S(\tau)$ for any finite τ. The theorem follows from the uniqueness assertion of the theorem on quasi-polyadic algebras.

THEOREM 4. *Let $\mathfrak{B} = (A, I, C, d)$ be a locally finite cylindric algebra of infinite degree. Let $\mathfrak{A} = (A, I, S, \exists, e)$ be the e-algebra arising from $\mathfrak{B}$ by means of Definition* 6 (*cf. Theorem* 2). *Let $\mathfrak{B}_1 = (A, I, C_1, d_1)$ be the*

cylindric algebra arising from $\mathfrak{A}$ *by means of Definition* 5 (*cf. Theorem* 1). *Then* $\mathfrak{B}=\mathfrak{B}_1$.

PROOF. From the definitions, we have $d_1=e=d$, and $C_1(k)=\exists(k)=C(k)$ for all $k\in I$.

BIBLIOGRAPHY

1. P. R. Halmos, *Polyadic Boolean algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 296–301.

2. ———, *Algebraic logic*, II, to appear in Fund. Math. in 1957.

3. A. Tarski and F. B. Thompson, *Some general properties of cylindric algebras*, Bull. Amer. Math. Soc. Abstract 58-1-85.

4. A. Tarski, *A representation theorem for cylindric algebras*, Bull. Amer. Math. Soc. Abstract 58-1-86.

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