SOME CONSEQUENCES OF THE APPROXIMATION THEOREM OF BING

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1. Introduction. In the following paragraphs we consider several unsolved problems in the topology of 3-space. Among the results is included a new formulation of properties characterizing tame curves in three-space without any explicit mention of polyhedral sets.

Three-space is denoted by \( R^3 \), the null-set by \( \Box \), and the set of points in \( A \) but not in \( B \) by \( A \setminus B \). The combinatorial boundary of chain \( E \) is \( \partial E \) and the set of points on such a boundary by \( | \partial E | \). The closure of a set \( A \) is denoted by \( \overline{A} \), or \( A \).

Theorem (Bing). Given a compact 2-manifold \( M \) of \( R^3 \) with or without boundaries, a compact subset \( D \) of \( R^3 \) and a positive number \( \epsilon \). There is a 2-manifold \( N \) and a homeomorphism \( f \) of \( M \) on \( N \) such that no point of \( M \) is moved more than \( \epsilon \) and \( N \) is locally polyhedral at each point of \( N \setminus D \). If \( D \neq \Box \), \( f \) may also be taken so that no point \( x \) is moved more than the minimum of \( \epsilon \) and the distance from \( x \) to \( D \). In particular, \( f \) is the identity map at all points of \( D \cap M \) and \( D \cap M = D \cap N \) [1].

2. The "orthogonal" disk problem. Some simple closed curves in \( R^3 \) have the property that, if \( J \) is the curve, there is a disk \( D \) whose boundary links \( J \) and such that \( J \) and \( D \) meet in a single point. This property is preserved by homeomorphisms acting on \( R^3 \). The class of curves having this property is known to include curves \( \{ J \} \) such that under some homeomorphism of \( R^3 \) on itself some sub-arc of \( J \) is mapped on a rectifiable arc [5]. By the use of his approximation theorem, R. H. Bing [2] has constructed an example of a curve that admits no orthogonal disk.

3. A new formulation of properties \( \varnothing \) and \( \mathcal{Q} \). A 1-manifold \( J \) is said to have property \( \varnothing \) if at each point \( x \) of \( J \) and for positive \( \epsilon \) there is a set \( K(x, \epsilon) \) of diameter less than \( \epsilon \) that is a topological 2-sphere whose interior contains \( x \), that meets \( J \) at a set of points whose cardinal does not exceed the order of \( x \) in \( J \), and such that \( K \) is locally polyhedral modulo \( J \).

To see that the locally polyhedral character of \( K \) may be omitted

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suppose $K$ has the other requirements and $\epsilon' < \epsilon = \text{diameter } K$. Using $M = K$, $D = J$ and $\epsilon'$ in the approximation theorem, then $N = K_1(x, \epsilon)$ has the original requirements.

A 1-manifold $J$ is said to have property $Q$ at $x$ if there is a disk $D_1$ such that $J \cap D_1$ is the closure of a neighborhood of $x$ in $J$ and $D_1$ is locally polyhedral at the points of $D_1 \setminus J$. To see that the locally polyhedral character of $D_1$ may be omitted, suppose $D_2$ has the requirements of $D_1$ apart from locally polyhedral character. If $M = D_2$ and $D = J$ and $\epsilon$ arbitrary in the approximation theorem we find a disk $N = D_1$ that is locally polyhedral at points of $N \setminus J$ and has the other requirements.

4. 1-cells contained in the interior of 2-cells. It has been known since 1921 when Antoine's thesis appeared that an arc in $R^3$ may be a subset of no topological 2-cell in $R^3$. Such an arc certainly does not lie on the boundary of a 2-cell and still less on the boundary of a 3-cell. (That a 2-cell in $R^3$ may be a subset of no topological 2-sphere and hence lie on the boundary of no 3-cell is implied by a result announced by Kapuano [7]). We limit ourselves here chiefly to the special case where the 1-cell lies interior (relative) to a 2-cell.

A fairly simple proof is now available that arcs exist lying on no 2-cell. Example 1.4 of Fox-Artin [3] is an example of a wild arc that is the union of two tame arcs, having only an end point in common. If $J$ is the set of points on this arc and $J_1$ is a sub-arc of $J$ whose interior contains the "wild point" of the imbedded arc, then $J_1$ is also wild. Now if $J$ were a part of the boundary of some disk $D$ in $R^3$, then $J_1$ has properties $\emptyset$ and $\mathcal{Q}$ (in the new form) and, by Theorem 6 of [6], would be tame. But $J_1$ is wild.

**Theorem.** Let $J$ be a closed 1-cell in $R^3$, then $J$ lies in the interior of a closed 2-cell if and only if $J$ lies on the boundary of some bounded, open, connected ulc subset of $R^3$.

If: Let $A$ be the bounded, open, connected subset of $R^3$ that is ulc and has $J$ on its boundary. By the results of Wilder, [8, Theorem 8.3, p. 311], every nondegenerate component of the boundary $M$ of $A$ is a classical 2-manifold. If $M_1$ is the component of $M$ determined by $J$, then, by triangulation of $M_1$, an open 2-cell $E$ in $M_1$ may be found such that $J$ is in Int $E$ (rel $M_1$).

Only if: If $J$ lies in the interior of a disk $E$, there is a sub-disk $E_1$ of $E$ such that $J$ is in $| \partial E_1 |$. Using the approximation theorem (with $D = J$, $M = E_1$) there is an "almost" polyhedral disk $N$ whose boundary contains $J$ that is locally polyhedral at points of $N \setminus J$. One may "inflate" $N$ to obtain a closed 3-cell $F$ that is locally polyhedral at
points $F \setminus J$ and $|\partial F|$ contains $J$. Then $J$ is contained in the closure of the interior of $F$ and $\text{Int } F$ is the required bounded, open, connected, ulc subset.

5. **A special case of the union of two locally tame sets.** It is known that locally tame sets are tame. However, the union of two tame sets having a single common point need not be tame. In this connection the following theorem is of interest.

**Theorem.** Let $J$ be a simple closed curve that is polyhedral and let $J$ be the boundary of any disk $F$ in $R^3$. Then $J$ is the boundary of a polyhedral disk.

(Thus $J$ is tame and unknotted).

**Proof.** If $J$ is the boundary of any disk $D$, then $J$ has the (new) property Q at each point. Since $J$ is polyhedral, it has property P at each point. By Theorem VII of [6] it is locally tamely imbedded, hence tamely imbedded. By Graeub [4, p. 39], this implies there is a polyhedral 2-cell of which $J$ is the boundary.

**References**


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