AN INTERPOLATION THEOREM FOR SUBLINEAR OPERATORS ON H_p SPACES

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The purpose of this paper is to extend an interpolation theorem for sublinear operators on L_p spaces, obtained by Calderón and Zygmund, to similar operators acting on H_p spaces, $p>0$ [1]. We begin by making a few definitions.

Let T be a mapping defined on a class of functions that are analytic in the interior of the unit circle $|w|<1$, taking values that are measurable functions on some measure space (M, μ). We say that T is *sublinear* if

(i) TF is uniquely defined if $F=F_1+F_2$ and TF_1 and TF_2 are defined;

(ii) For any constant k, $T(kF)$ is defined if TF is defined;

(iii) $|T(F_1+F_2)| \leqq |TF_1|+|TF_2|$, $|T(kF)|=|k|\cdot|T(F)|$ almost everywhere.

We shall also say that T is of *type* (a, b), $a, b>0$, if the domain of T includes H_a and there exists a constant M, independent of F in H_a, such that $\|TF\|_b \leqq M\cdot\|F\|_a$ (the norm on the left is the L_b norm and the norm on the right is the H_a norm).

For the results in the theory of H_p spaces we shall use, we refer the reader to [2].

THEOREM. *Let T be a sublinear operator of types $(1/\alpha_1, 1/\beta_1)$ and $(1/\alpha_2, 1/\beta_2)$ with constants M_1 and M_2 respectively, where $1/\alpha_i>0$, $1/\beta_i \geqq 1$ $(i=1, 2)$. Let $\alpha=(1-t_0)\alpha_1+t_0\alpha_2$ and $\beta=(1-t_0)\beta_1+t_0\beta_2$, where $0\leqq t_0\leqq 1$. Then there exists an absolute constant, K, such that*

$$\|TF\|_{1/\beta} \leqq KM_1^{(1-t_0)}M_2^{t_0}\|F\|_{1/\alpha}$$

for all F in $H_{1/\alpha}$ and, thus, T is of type $(1/\alpha, 1/\beta)$.

We remark that the above theorem in the special case where T is linear is known [3].

Let us assume that $\alpha_1\leqq\alpha\leqq\alpha_2$. Let n be an integer such that n/α_1, $n/\alpha_2>1$. In the proof we shall suppose that $n=2$, the proof being easily extensible to the general case.

Let $f=\sum_k c_k\epsilon_k\chi_k$, where $c_k>0$, $|\epsilon_k|=1$ and the χ_k's are a finite number of characteristic functions of mutually disjoint measurable subsets, E_k, of $[0, 2\pi]$. Let

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$$F(w) = 1/2\pi \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} f(t)dt. \tag{1}$$

It is well known that F is in H_p for all $p>0$ and, thus, F^2 is in H_p for all $p>0$. Hence, $T(F^2)$ is well defined. We are going to make use of the fact that

$$\|T(F^2)\|_{1/\beta} = \sup \left\{ \int_M |T(F^2)|\, g d\mu;\, g \text{ simple}, g \geqq 0, \|g\|_{1/1-\beta} = 1 \right\}. \tag{2}$$

We make the further assumption that $\|f\|_{2/\alpha} = 1$.

Set $g = \sum_i c_i' \chi_i'$ where $c_i' > 0$ and χ_i' is the characteristic function of the measurable set E_i' of M, and suppose that $\|g\|_{1/1-\beta} = 1$. We shall now estimate

$$\int_M |T(F^2)|\, g d\mu. \tag{3}$$

This estimate will then give us a bound for $\|T(F^2)\|_{1/\beta}$.

Let $z = x + iy$ be any complex number and define

$$F_z(w) = 1/2\pi \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} |f(t)|^{\alpha(z)/\alpha} e^{i \arg f(t)} dt$$

where $\alpha(z) = (1-z)\alpha_1 + z\alpha_2$ and $\beta(z) = (1-z)\beta_1 + z\beta_2$. We now consider

$$\Phi(z) = \int_M |T(F_z^2)|\, g^{1-\beta(x)/1-\beta} d\mu. \tag{4}$$

Clearly, when $z = t_0$, $\Phi(z)$ reduces to (3).

Our plan is to show that $\log \Phi(z)$ is a continuous subharmonic function that is bounded in every vertical strip in the plane. An application of the three line theorem for such functions will then give us

$$\log \Phi(t_0 + iy) \leqq (1 - t_0) \log \max_y \Phi(iy) + t_0 \log \max_y \Phi(1 + iy). \tag{5}$$

But, making use of Hölder's inequality and M. Riesz's inequality [2], which implies that if h is in $L_p(0, 2\pi)$, $p>1$, and

$$H(w) = 1/2\pi \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} h(t)dt$$

then there exists a constant, A_p, independent of h, such that $\|H\|_p \leqq A_p \cdot \|h\|_p$, we have, on the line $x=0$

$$\Phi(iy) \leqq \|TF_{iy}^2\|_{1/\beta_1} \cdot \|g^{1-\beta(0)/1-\beta}\|_{1/1-\beta_1} \leqq M_1 \|F_{iy}^2\|_{1/\alpha_1} \cdot 1 = M_1 \|F_{iy}\|_{2/\alpha_1}^2$$

$$\leqq M_1 \cdot A_{2/\alpha_1}^2 \| \, | f(t) |^{\alpha(iy)/\alpha} \|_{2/\alpha_1}^2 = M_1 A_{2/\alpha_1}^2$$

since $\|f\|_{2/\alpha}=1$ and $\|g\|_{1/1-\beta}=1$ and $\{ \, | \, |f|^{\alpha(iy)/\alpha} | \, \}^{2/\alpha_1} = \{ \, |f|^{\alpha} \, \}^{2/\alpha_1} = |f|^{2/\alpha}$ and, similarly, $\{ \, | \, (g)^{1-\beta(0)/1-\beta} | \, \}^{1/1-\beta_1} = |g|^{1/1-\beta}$.

Likewise, on the line $x=1$ we have $\Phi(1+iy) \leqq M_2 A_{2/\alpha_2}^2$.

Thus, letting $y=0$ in (5), since g was an arbitrary simple function satisfying $\|g\|_{1/1-\beta}=1$, we obtain, upon comparing with (2)

$$\|T(F^2)\|_{1/\beta} \leqq K M_1^{(1-t_0)} M_2^{t_0} \tag{6}$$

where $K = \{ A_{2/\alpha_1}^{(1-t_0)} A_{2/\alpha_2}^{t_0} \}^2$.

The last inequality is the conclusion of the theorem for the special case of functions in $H_{1/\alpha}$ that are squares of functions of the type (1) with $\|f\|_{2/\alpha}=1$. We will show later on, however, that we can deduce the theorem from (6).

The fact that $\log \Phi(z)$ is bounded in every vertical strip in the plane is immediate. For expanding (4) and using (iii) we obtain

$$\Phi(z) \leqq \sum_l c_l'^{1-\beta(x)/1-\beta} \int_{E_l'} \sum_{n,m} (c_n c_m)^{\alpha(x)/\alpha} | \, T(\Delta_n \cdot \Delta_m) \, | \, d\mu,$$

where

$$\Delta_s(w) = 1/2\pi \int_0^{2\pi} \frac{e^{it}+w}{e^{it}-w} \chi_s(t) dt.$$

We now proceed to show that $\log \Phi(z)$ is continuous. We have

$$\Phi(z) = \sum_l \int_{E_l'} c_l'^{1-\beta(x)/1-\beta} | \, T(F_z^2) \, | \, d\mu = \sum_l \int_{E_l'} | \, T c_l'^{1-\beta(x)/1-\beta}(F_z^2) \, | \, d\mu.$$

We shall investigate each of these summands. Let

$$\psi_{l,z} = \psi_z = c_l'^{1-\beta(x)/1-\beta}(F_z^2)$$

and $\Psi_l(z) = \Psi(z) = \int_{E_l'} | \, T\psi_z | \, d\mu$.

It follows immediately from (i), (ii), and (iii) that if TF_1 and TF_2 are defined so is $T(F_1 - F_2)$ and we have

$$| \, | \, TF_1 | \, - \, | \, TF_2 | \, | \leqq | \, T(F_1 - F_2) \, | \, . \tag{7}$$

It clearly suffices to show that $\Psi(z)$ is continuous. Using (7) and Hölder's inequality, we obtain

$$
\begin{aligned}
|\Psi(z) - \Psi(z+\Delta z)| &= \left| \int_{E_l'} |T\psi_{z+\Delta z}|\, d\mu - \int_{E_l'} |T\psi_z|\, d\mu \right| \\
&\leqq \int_{E_l'} |T(\psi_{z+\Delta z} - \psi_z)|\, d\mu \\
&\leqq \|T(\psi_{z+\Delta z} - \psi_z)\|_{1/\beta_1} \cdot \{\mu(E_l')\}^{1/1-\beta_1} \\
&\leqq M_1 \cdot \|\psi_{z+\Delta z} - \psi_z\|_{1/\alpha_1} \cdot \{\mu(E_l')\}^{1/1-\beta_1}.
\end{aligned}
$$

Thus, if we can show that $\|\psi_{z+\Delta z} - \psi_z\|_{1/\alpha_1} \to 0$ as $\Delta z \to 0$ we have the desired continuity. We proceed to do this:

$$
|\psi_{z+\Delta z} - \psi_z| = \left| \{c_l'^{1-\beta(z+\Delta z)/1-\beta}\} F_{z+\Delta z}^2 - \{c_l'^{1-\beta(z)/1-\beta}\} F_z^2 \right| \leqq A + B,
$$

where

$$
A = \left| \{c_l'^{1-\beta(z+\Delta z)/1-\beta}\} (F_{z+\Delta z}^2 - F_z^2) \right|
$$

and

$$
B = \left| (c_l'^{1-\beta(z+\Delta z)/1-\beta} - c_l'^{1-\beta(z)/1-\beta}) F_z^2 \right|.
$$

If $1/\alpha_1 \leqq 1$ we thus have

$$
\|\psi_{z+\Delta z} - \psi_z\|_{1/\alpha_1}^{1/\alpha_1} \leqq \|A\|_{1/\alpha_1}^{1/\alpha_1} + \|B\|_{1/\alpha_1}^{1/\alpha_1}. \tag{8}
$$

(If $1/\alpha_1 > 1$ we do not take the $1/\alpha_1$ powers of the norms and use Minkowski's inequality instead of (8) and the remaining part of the proof is essentially the same.)

We have, using Schwarz's inequality:

$$
\begin{aligned}
\|A\|_{1/\alpha_1} &= \left| c_l'^{1-\beta(z+\Delta z)/1-\beta} \right| \cdot \|F_{z+\Delta z}^2 - F_z^2\|_{1/\alpha_1} \\
&\leqq \left| c_l'^{1-\beta(z+\Delta z)/1-\beta} \right| \cdot \|F_{z+\Delta z} - F_z\|_{2/\alpha_1} \cdot \|F_{z+\Delta z} + F_z\|_{2/\alpha_1}.
\end{aligned}
$$

Using M. Riesz's inequality we have that the last terms are dominated by

$$
\begin{aligned}
A_{2/\alpha_1}^2 \left| c_l'^{1-\beta(z+\Delta z)/1-\beta} \right| &\cdot \left\| \, |f(t)|^{\alpha(z+\Delta z)/\alpha} - |f(t)|^{\alpha(z)/\alpha} \right\|_{2/\alpha_1} \\
&\cdot \left\| \, |f(t)|^{\alpha(z+\Delta z)/\alpha} + |f(t)|^{\alpha(z)/\alpha} \right\|_{2/\alpha_1}.
\end{aligned}
$$

But $|f(t)|^{\alpha(z+\Delta z)/\alpha} - |f(t)|^{\alpha(z)/\alpha}$ is 0 outside $\cup E_k$ and tends to 0 uniformly in t there as $\Delta z \to 0$. Therefore,

$$
\left\| \, |f(t)|^{\alpha(z+\Delta z)/\alpha} - |f(t)|^{\alpha(z)/\alpha} \right\|_{2/\alpha_1} \to 0 \text{ as } \Delta z \to 0.
$$

On the other hand, both $|c_l'^{1-\beta(z+\Delta z)/1-\beta}|$ and $\| |f(t)|^{\alpha(z+\Delta z)/\alpha} + |f(t)|^{\alpha(z)/\alpha}\|_{2/\alpha_1}$ are bounded for Δz small. This shows that $\|A\|_{1/\alpha_1} \to 0$ as $\Delta z \to 0$. We also have

$$\|B\|_{1/\alpha_1} = \left| c_l'^{1-\beta(z+\Delta z)/1-\beta} - c_l'^{1-\beta(z)/1-\beta} \right| \cdot \|F_z^2\|_{1/\alpha_1} \to 0$$

as $\Delta z \to 0$.

Hence, we have shown that $\|\psi_{z+\Delta z} - \psi_z\|_{1/\alpha_1} \to 0$ as $\Delta z \to 0$ and, thus, $\Psi(z)$ is continuous.

We shall now show that $\log \Phi(z)$ is subharmonic. It is sufficient to show that if $h(z)$ is any harmonic function then $\Phi(z)e^{h(z)}$ is subharmonic. To achieve this end it is sufficient to show that $\Psi_l(z)e^{h(z)} = \Psi(z)e^{h(z)}$ is subharmonic, for then

$$\Phi(z)e^{h(z)} = \sum_l \Psi_l(z)e^{h(z)}$$

being a sum of subharmonic functions is subharmonic. Thus we fix $h(z)$, an arbitrary harmonic function, and denote by $H(z)$ an analytic function whose real part is $h(z)$. Since the problem is local, we may consider h and H in a given circle. We write

$$\psi_z^* = \psi_z e^{H(z)} \text{ and } \Psi^*(z) = \Psi(z)e^{h(z)} = \int_{E_l'} |T\psi_z^*| \, d\mu.$$

Fix z and let $\rho > 0$, and denote by $z_1, z_2, \cdots, z_p$ a collection of points equally spaced over the circumference of the circle with center z and radius ρ. We have

$$\psi_z^*(w) = \lim_{q \to \infty} 1/q \sum_{j=1}^{q} \psi_{z_j}^*(w) \tag{9}$$

since $\psi_z^*(w)$ is analytic for each w in the unit circle. This is easily seen since

$$\psi_z(w) = c_l'^{1-\beta(z)/1-\beta} \left\{ \sum_k c_k^{\alpha(z)/\alpha} \epsilon_k \Delta_k(w) \right\}^2$$

is clearly analytic for each w. It will be necessary for our purpose to analyze further the convergence in (9). More precisely, we will show

$$\|\psi_z^*(w) - 1/q \sum_{j=1}^{q} \psi_{z_j}^*(w)\|_{1/\alpha_1} \to 0 \text{ as } q \to \infty. \tag{10}$$

Suppose that $1/\alpha_1 \leqq 1$ (if $1/\alpha_1 > 1$ we do not take the $1/\alpha_1$ powers and, as before, use Minkowski's inequality, the rest of the proof being the same). We have

$$\left\| \psi_z^*(w) - 1/q \sum_{j=1}^{q} \psi_{z_j}^*(w) \right\|_{1/\alpha_1}^{1/\alpha_1}$$

$$= \left\| e^{H(z)} c_l'^{1-\beta(z)/1-\beta} F_z^2 - 1/q \sum_{j=1}^{q} e^{H(z_j)} c_l'^{1-\beta(z_j)/1-\beta} F_{z_j}^2 \right\|_{1/\alpha_1}^{1/\alpha_1}$$

$$= \left\| e^{H(z)} c_l'^{1-\beta(z)/1-\beta} \sum_{n,m} (c_n c_m)^{\alpha(z)/\alpha} \epsilon_n \epsilon_m \Delta_n(w) \Delta_m(w) \right.$$

$$\left. - 1/q \sum_{j=1}^{q} e^{H(z_j)} c_l'^{1-\beta(z_j)/1-\beta} \sum_{n,m} (c_n c_m)^{\alpha(z_j)/\alpha} \epsilon_n \epsilon_m \Delta_n(w) \Delta_m(w) \right\|_{1/\alpha_1}^{1/\alpha_1}$$

$$\leqq \sum_{n,m} \left\{ \left| e^{H(z)} c_l'^{1-\beta(z)/1-\beta} (c_n c_m)^{\alpha(z)/\alpha} \epsilon_n \epsilon_m \right.\right.$$

$$\left.\left. - 1/q \sum_{j=1}^{q} e^{H(z_j)} c_l'^{1-\beta(z_j)/1-\beta} (c_n c_m)^{\alpha(z_j)/\alpha} \epsilon_n \epsilon_m \right| \cdot \left\| \Delta_n(w) \Delta_m(w) \right\|_{1/\alpha_1} \right\}^{1/\alpha_1}.$$

But each of the terms

$$e^{H(z)} c_l'^{1-\beta(z)/1-\beta} (c_n c_m)^{\alpha(z)/\alpha} - 1/q \sum_{j=1}^{q} e^{H(z_j)} c_l'^{1-\beta(z_j)/1-\beta} (c_n c_m)^{\alpha(z_j)/\alpha}$$

converge to 0 as $q \to \infty$. This proves (10).

We thus have

$$\int_{E_l'} \left| T\left(\psi_z^* - 1/q \sum_1^q \psi_{z_j}^*\right) \right| d\mu \leqq \left\| T\left(\psi_z^* - 1/q \sum_1^q \psi_{z_j}^*\right) \right\|_{1/\beta_1} \{\mu(E_l')\}^{1/1-\beta_1}$$

$$\leqq M_1 \left\| \psi_z^* - 1/q \sum_1^q \psi_{z_j}^* \right\|_{1/\alpha_1} \cdot \{\mu(E_l')\}^{1/1-\beta_1} \to 0 \text{ as } q \to \infty.$$

A fortiori, we have

$$e_q = \int_{E_l'} \left| \, |T\psi_z^*| - \left| T\left(1/q \sum_1^q \psi_{z_j}^*\right) \right| \right| d\mu \to 0, \text{ as } q \to \infty.$$

Thus, since

$$\int_{E_l'} |T\psi_q^*| \, d\mu \leqq e_q + 1/q \sum_1^q \int_{E_l'} |T\psi_{z_j}^*| \, d\mu$$

we obtain

$$\Psi^*(z) \leqq \lim_{q \to \infty} 1/q \sum_1^q \Psi^*(z_j) = 1/2\pi \int_0^{2\pi} \Psi^*(z + \rho e^{it}) dt.$$

We have thus proved that $\log \Phi(z)$ is subharmonic. Thus our next task is to show that (6) implies the theorem.

Suppose that F is defined as in equation (1) where f is a simple function, but we no longer assume that $\|f\|_{2/\alpha}=1$. We then have $\|f\|_{2/\alpha} \leqq \|F\|_{2/\alpha}$. Thus

$$\begin{aligned}\|T(F^2/\|F\|^2_{2/\alpha})\|_{1/\beta} &= 1/\|F\|^2_{2/\alpha}\cdot\|T(F^2)\|_{1/\beta} \leqq 1/\|f\|^2_{2/\alpha}\cdot\|T(F^2)\|_{1/\beta} \\ &= \|T(F^2/\|f\|^2_{2/\alpha})\|_{1/\beta} \leqq KM_1^{(1-t_0)}M_2^{t_0}\end{aligned}$$

since $F/\|f\|_{2/\alpha}$ is a function of type (1) for which this inequality was proved. But

$$\|F\|^2_{2/\alpha} = \|F^2\|_{1/\alpha}.$$

We thus have, using the last part of (iii)

$$\|T(F^2)\|_{1/\beta} \leqq KM_1^{(1-t_0)}M_2^{t_0}\|F^2\|_{1/\alpha} \tag{11}$$

for any function of type (1) where f is simple.

Now let G be a function in $H_{1/\alpha}$ that is the square of a function F in $H_{2/\alpha}$. Let f be a function in $L^{2/\alpha}(0, 2\pi)$ such that $F(w) = 1/2\pi\int_0^{2\pi}((e^{it}+w/e^{it}-w))f(t)dt$ (i.e. f is the real part of $F(e^{it})$). Let $\{f_n\}$ be a sequence of simple functions such that $\|f_n-f\|_{2/\alpha}\to 0$ as $n\to\infty$ and set $F_n(w)=1/2\pi\int_0^{2\pi}((e^{it}+w)/(e^{it}-w))f_n(t)dt$. Then $F^2-F_n^2$ is in $H_{1/\alpha}\subset H_{1/\alpha_2}$ (since $1/\alpha \geqq 1/\alpha_2$). Thus, in particular, F^2 is a sum of a function F_n in H_{1/α_1} and a function $F^2-F_n^2$ in H_{1/α_2}, showing, by (i), that $T(F^2)$ is defined. We also have

$$\big|\,|TF^2| - |TF_n^2|\,\big| \leqq |T(F^2-F_n^2)|. \tag{12}$$

But $\|T(F^2-F_n^2)\|_{1/\beta_2} \leqq M_2\|F^2-F_n^2\|_{1/\alpha_2} \leqq M_2\|F-F_n\|_{2/\alpha_2}\cdot\|F+F_n\|_{2/\alpha_2} \leqq M_2A^2_{2/\alpha_2}\|f-f_n\|_{2/\alpha_2}\cdot\|f+f_n\|_{2/\alpha_2} \leqq CM_2A^2_{2/\alpha_2}\|f-f_n\|_{2/\alpha}\cdot\|f+f_n\|_{2/\alpha}$ where C does not depend on the functions f and f_n but is such that $\|g\|_{2/\alpha_2} \leqq C^{1/2}\|g\|_{2/\alpha}$ for any g in $L_{2/\alpha}(0, 2\pi)$ (C exists because the measure space is finite). The last factor is bounded since $\|f_n-f\|_{2/\alpha}\to 0$ and, by this very last fact, the entire right-hand side of the inequality goes to 0 as $n\to\infty$. Thus there exists a subsequence $|T(F^2-F^2_{n_k})|\to 0$ almost everywhere and, consequently, $|T(F^2_{n_k})|\to|T(F^2)|$ almost everywhere, by (12). By Fatou's lemma, therefore, we have

$$\begin{aligned}\int |T(F^2)|^{1/\beta}d\mu &\leqq \liminf \int |T(F^2_{n_k})|^{1/\beta}d\mu \\ &\leqq \liminf \{KM_1^{(1-t_0)}M_2^{t_0}\|F^2_{n_k}\|_{1/\alpha}\}^{1/\beta} \\ &= \{KM_1^{(1-t_0)}M_2^{t_0}\|F^2\|_{1/\alpha}\}^{1/\beta}.\end{aligned}$$

We have shown

$$\|T(F^2)\|_{1/\beta} \leqq KM_1^{(1-t_0)}M_2^{t_0}\|F^2\|_{1/\alpha} \tag{13}$$

for any function F in $H_{2/\alpha}$.

But any function G in $H_{1/\alpha}$ is a sum of two functions, G_1 and G_2, with no zeros, both in $H_{1/\alpha}$, such that $\|G_1\|_{1/\alpha}=\|G\|_{1/\alpha}$ and $\|G_2\|_{1/\alpha} \leqq 2\|G\|_{1/\alpha}$. This is easily seen by the Blaschke product decomposition, $G=BG_1$ where B is the Blaschke product which satisfies $|B(z)|<1$ if $|z|<1$, and G_1 has no zeros. Thus, $G=(B-1)G_1+G_1$ is the desired decomposition with $G_2=(B-1)G_1$. The norm inequalities follow from the fact that $\|G_1\|_{1/\alpha}=\|G\|_{1/\alpha}$ and $|B-1|\leqq 2$ in the unit circle. Since G_1 and G_2 have no zeros they can be written as the squares of their square roots, and, clearly, these square roots are in $H_{2/\alpha}$. Thus, using (iii)

$$\begin{aligned}\|TG\|_{1/\beta} = \|T(G_1+G_2)\|_{1/\beta} &\leqq \|\,|TG_1| + |TG_2|\,\|_{1/\beta} \leqq \|TG_1\|_{1/\beta} + \|TG_2\|_{1/\beta} \\ &\leqq KM_1^{(1-t_0)}M_2^{t_0}\{\|G_1\|_{1/\alpha} + \|G_2\|_{1/\alpha}\} \\ &\leqq KM_1^{(1-t_0)}M_2^{t_0}\{\|G\|_{1/\alpha} + 2\|G\|_{1/\alpha}\} = 3KM_1^{(1-t_0)}M_2^{t_0}\|G\|_{1/\alpha}.\end{aligned}$$

This proves the theorem.

References

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