AN INTERPOLATION THEOREM FOR SUBLINEAR OPERATORS ON $H_p$ SPACES

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The purpose of this paper is to extend an interpolation theorem for sublinear operators on $L_p$ spaces, obtained by Calderón and Zygmund, to similar operators acting on $H_p$ spaces, $p > 0$ [1]. We begin by making a few definitions.

Let $T$ be a mapping defined on a class of functions that are analytic in the interior of the unit circle $|w| < 1$, taking values that are measurable functions on some measure space $(M, \mu)$. We say that $T$ is sublinear if

(i) $TF$ is uniquely defined if $F = F_1 + F_2$ and $TF_1$ and $TF_2$ are defined;

(ii) For any constant $k$, $T(kF)$ is defined if $TF$ is defined;

(iii) $|T(F)| \leq |TF_1| + |TF_2|$, $|T(kF)| = |k| \cdot |T(F)|$ almost everywhere.

We shall also say that $T$ is of type $(a, b), a, b > 0$, if the domain of $T$ includes $H_a$ and there exists a constant $M$, independent of $F$ in $H_a$, such that $\|TF\|_b \leq M \cdot \|F\|_a$ (the norm on the left is the $L_b$ norm and the norm on the right is the $H_a$ norm).

For the results in the theory of $H_p$ spaces we shall use, we refer the reader to [2].

Theorem. Let $T$ be a sublinear operator of types $(1/\alpha_1, 1/\beta_1)$ and $(1/\alpha_2, 1/\beta_2)$ with constants $M_1$ and $M_2$ respectively, where $1/\alpha_i > 0$, $1/\beta_i \geq 1$ ($i = 1, 2$). Let $\alpha = (1 - t_0)\alpha_1 + t_0\alpha_2$ and $\beta = (1 - t_0)\beta_1 + t_0\beta_2$, where $0 \leq t_0 \leq 1$. Then there exists an absolute constant, $K$, such that

$$\|TF\|_{1/\beta} \leq KM_1^{(1-t_0)}M_2^{t_0}\|F\|_{1/\alpha}$$

for all $F$ in $H_{1/\alpha}$ and, thus, $T$ is of type $(1/\alpha, 1/\beta)$.

We remark that the above theorem in the special case where $T$ is linear is known [3].

Let us assume that $\alpha_1 \leq \alpha \leq \alpha_2$. Let $n$ be an integer such that $n/\alpha_1, n/\alpha_2 > 1$. In the proof we shall suppose that $n = 2$, the proof being easily extensible to the general case.

Let $f = \sum_k c_k \varepsilon_k \chi_k$, where $c_k > 0$, $|\varepsilon_k| = 1$ and the $\chi_k$'s are a finite number of characteristic functions of mutually disjoint measurable subsets, $E_k$, of $[0, 2\pi]$. Let

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\[ F(w) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + w}{e^{it} - w} f(t) \, dt. \]

It is well known that \( F \) is in \( H_p \) for all \( p > 0 \) and, thus, \( F^2 \) is in \( H_p \) for all \( p > 0 \). Hence, \( T(F^2) \) is well defined. We are going to make use of the fact that

\[ ||T(F^2)||_{1/\beta} = \sup \left\{ \int_{M} |T(F^2)| g \, d\mu; g \text{ simple, } g \geq 0, ||g||_{1/1-\beta} = 1 \right\}. \]

We make the further assumption that \( ||f||_{1/\alpha} = 1 \).

Set \( g = \sum i c_i \chi'_i \) where \( c'_i > 0 \) and \( \chi'_i \) is the characteristic function of the measurable set \( E'_i \) of \( M \), and suppose that \( ||g||_{1/1-\beta} = 1 \). We shall now estimate

\[ \int_{M} |T(F^2)| g \, d\mu. \]

This estimate will then give us a bound for \( ||T(F^2)||_{1/\beta} \).

Let \( z = x + iy \) be any complex number and define

\[ F_z(w) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + w}{e^{it} - w} f(t) |\alpha(t)/\alpha e^{i\beta} f(t)| \, dt \]

where \( \alpha(z) = (1 - z)\alpha_1 + z\alpha_2 \) and \( \beta(z) = (1 - z)\beta_1 + z\beta_2 \). We now consider

\[ \Phi(z) = \int_{M} |T(F_z)| g^{1-\beta(z)/1-\beta} \, d\mu. \]

Clearly, when \( z = t_0 \), \( \Phi(z) \) reduces to (3).

Our plan is to show that \( \log \Phi(z) \) is a continuous subharmonic function that is bounded in every vertical strip in the plane. An application of the three line theorem for such functions will then give us

\[ \log \Phi(t_0 + iy) \leq (1 - t_0) \log \max_{v} \Phi(iy) + t_0 \log \max_{v} \Phi(1 + iy). \]

But, making use of Hölder's inequality and M. Riesz's inequality [2], which implies that if \( h \) is in \( L_p(0, 2\pi) \), \( p > 1 \), and

\[ H(w) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + w}{e^{it} - w} h(t) \, dt \]

then there exists a constant, \( A_p \), independent of \( h \), such that \( ||H||_p \leq A_p \cdot ||h||_p \), we have, on the line \( x = 0 \)
\[ \Phi(iy) \leq \| T F_{i0} \|_{1/\beta} \| g^{1-\beta(0)/1-\beta} \|_{1/1-\beta_1} \leq M_1 \| F_{i0} \|_{1/\alpha_1} \cdot 1 = M_1 \| F_{i0} \|_{2/\alpha_1}^2 \]
\[ \leq M_1 \cdot A_{2/\alpha_1}^2 \| f(t) \|_{2/\alpha_1}^{\alpha(iy)/\alpha} \| g \|_{2/\alpha_1} = M_1 A_{2/\alpha_1}^2 \]

since \( \| f \|_{2/\alpha} = 1 \) and \( \| g \|_{1/1-\beta} = 1 \) and \( \{ \| f \|^{\alpha(iy)/\alpha} \}^2/\alpha_1 = \{ \| f \| \}^2/\alpha_1 = \| f \|_{2/\alpha} \) and, similarly, \( \{ \| g \|_{1/1-\beta} \}^2/\alpha_1 = \| g \|_{1/1-\beta} \).

Likewise, on the line \( x = 1 \) we have \( \Phi(1 + iy) \leq M_2 A_{2/\alpha_2}^2 \).

Thus, letting \( y = 0 \) in (5), since \( g \) was an arbitrary simple function satisfying \( \| g \|_{1/1-\beta} = 1 \), we obtain, upon comparing with (2)

\[ ||W||_{1/\alpha} \leq KM_1^{(1-t_0)/2} M_2^{t_0} \]

where \( K = \{ A_{2/\alpha_1}^{1-t_0} A_{2/\alpha_2}^{t_0} \}^2 \).

The last inequality is the conclusion of the theorem for the special case of functions in \( H_{1/\alpha} \) that are squares of functions of the type (1) with \( \| f \|_{2/\alpha} = 1 \). We will show later on, however, that we can deduce the theorem from (6).

The fact that \( \log \Phi(z) \) is bounded in every vertical strip in the plane is immediate. For expanding (4) and using (iii) we obtain

\[ \Phi(z) \leq \sum_i \sum_{n,m} c_i^{1-\beta(z)/1-\beta} \int_{E_i} c_n c_m^{\alpha(z)/\alpha} \| T(\Delta_n \cdot \Delta_m) \| \, d\mu, \]

where

\[ \Delta_s(w) = 1/2\pi \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} x_s(t) \, dt. \]

We now proceed to show that \( \log \Phi(z) \) is continuous. We have

\[ \Phi(z) = \sum_i \int_{E_i} c_i^{1-\beta(z)/1-\beta} \| T(F^2_i) \| \, d\mu = \sum_i \int_{E_i} c_i^{1-\beta(z)/1-\beta} (F^2_i) \, d\mu. \]

We shall investigate each of these summands. Let

\[ \Psi(z) = \phi(z) = c_i^{1-\beta(z)/1-\beta} (F^2_i) \]

and \( \Psi(z) = \Psi(z) = \int_{E_i} |T \psi(z)| \, d\mu. \)

It follows immediately from (i), (ii), and (iii) that if \( TF_1 \) and \( TF_2 \) are defined so is \( T(F_1 - F_2) \) and we have

\[ ||T(F_1 - F_2)|| \leq ||T(F_1) - T(F_2)||. \]

It clearly suffices to show that \( \Psi(z) \) is continuous. Using (7) and Hölder's inequality, we obtain
Thus, if we can show that $\|\psi_{s+\Delta s} - \psi_s\|_{1/\alpha_1} \to 0$ as $\Delta s \to 0$ we have the desired continuity. We proceed to do this:

$$\|\psi_{s+\Delta s} - \psi_s\| = \left\| \left\{ c_{i}^{1-\beta(s+\Delta s)/1-\beta} \right\} F_{s+\Delta s} - \left\{ c_{i}^{1-\beta(s)/1-\beta} \right\} F_{s} \right\| \leq A + B,$$

where

$$A = \left\| \left\{ c_{i}^{1-\beta(s+\Delta s)/1-\beta} \right\} (F_{s+\Delta s} - F_{s}) \right\|$$

and

$$B = \left\| \left( c_{i}^{1-\beta(s+\Delta s)/1-\beta} - c_{i}^{1-\beta(s)/1-\beta} \right) F_{s} \right\|.$$

If $1/\alpha_1 \leq 1$ we thus have

$$\|\psi_{s+\Delta s} - \psi_s\|_{1/\alpha_1}^{1/\alpha_1} \leq \|A\|_{1/\alpha_1}^{1/\alpha_1} + \|B\|_{1/\alpha_1}^{1/\alpha_1}.$$  

(If $1/\alpha_1 > 1$ we do not take the $1/\alpha_1$ powers of the norms and use Minkowski’s inequality instead of (8) and the remaining part of the proof is essentially the same.)

We have, using Schwarz’s inequality:

$$\|A\|_{1/\alpha_1} = c_{i}^{1-\beta(s+\Delta s)/1-\beta} \cdot \|F_{s+\Delta s} - F_{s}\|_{1/\alpha_1} \leq c_{i}^{1-\beta(s+\Delta s)/1-\beta} \cdot \|F_{s+\Delta s} - F_{s}\|_{2/\alpha_1} \cdot \|F_{s+\Delta s} + F_{s}\|_{2/\alpha_1}.$$  

Using M. Riesz’s inequality we have that the last terms are dominated by

$$A_{2/\alpha_1} c_{i}^{1-\beta(s+\Delta s)/1-\beta} \cdot \|f(t)\|_{1/\alpha_1}^{\alpha(s+\Delta s)/\alpha} - \|f(t)\|_{1/\alpha_1}^{\alpha(s)/\alpha} \cdot \|f(t)\|_{2/\alpha_1}^{\alpha(s)/\alpha} + \|f(t)\|_{2/\alpha_1}^{\alpha(s)/\alpha}.$$

But $|f(t)|^{\alpha(s+\Delta s)/\alpha} - |f(t)|^{\alpha(s)/\alpha}$ is 0 outside $UE_k$ and tends to 0 uniformly in $t$ there as $\Delta s \to 0$. Therefore,

$$\|f(t)\|_{2/\alpha_1}^{\alpha(s+\Delta s)/\alpha} - |f(t)|_{2/\alpha_1}^{\alpha(s)/\alpha} \to 0 \text{ as } \Delta s \to 0.$$
On the other hand, both $|c_i^{1-\beta(z+\Delta z)/1-\beta}|$ and $\|f(t)\|_{\alpha(z+\Delta z)/\alpha} + |f(t)|_{\alpha(z)/\alpha}^{2/\alpha_1}$ are bounded for $\Delta z$ small. This shows that $\|A\|_{1/\alpha_1} \to 0$ as $\Delta z \to 0$. We also have

$$\|B\|_{1/\alpha_1} = \left| c_i^{1-\beta(z+\Delta z)/1-\beta} - c_i^{1-\beta(z)/1-\beta} \right| \cdot \|F\|_{1/\alpha_1} \to 0$$

as $\Delta z \to 0$.

Hence, we have shown that $\|\Psi_{z+\Delta z} - \Psi_{z}\|_{1/\alpha_1} \to 0$ as $\Delta z \to 0$ and, thus, $\Psi(z)$ is continuous.

We shall now show that log $\Phi(z)$ is subharmonic. It is sufficient to show that if $h(z)$ is any harmonic function then $\Phi(z)e^{h(z)}$ is subharmonic. To achieve this end it is sufficient to show that $\Psi_{l}(z)e^{h(z)} = \Psi(z)e^{h(z)}$ is subharmonic, for then

$$\Phi(z)e^{h(z)} = \sum_{i} \Psi_{l}(z)e^{h(z)}$$

being a sum of subharmonic functions is subharmonic. Thus we fix $h(z)$, an arbitrary harmonic function, and denote by $H(z)$ an analytic function whose real part is $h(z)$. Since the problem is local, we may consider $h$ and $H$ in a given circle. We write

$$\Psi_{z} = \Psi_{z}e^{H(z)} \quad \text{and} \quad \Psi^{*}(z) = \Psi(z)e^{h(z)} = \int_{E_{1}} T\Psi_{z} d\mu.$$ 

Fix $z$ and let $\rho > 0$, and denote by $z_1, z_2, \cdots, z_p$ a collection of points equally spaced over the circumference of the circle with center $z$ and radius $\rho$. We have

$$\Psi_{z}^{*}(w) = \lim_{q \to \infty} \frac{1}{q} \sum_{j=1}^{q} \Psi_{z_j}^{*}(w)$$

since $\Psi_{z}^{*}(w)$ is analytic for each $w$ in the unit circle. This is easily seen since

$$\Psi_{z}(w) = \left( \sum_{k} c_k a(z)/\alpha \epsilon_k \Delta_k(w) \right)^{2}$$

is clearly analytic for each $w$. It will be necessary for our purpose to analyze further the convergence in (9). More precisely, we will show

$$\|\Psi_{z}^{*}(w) - \frac{1}{q} \sum_{j=1}^{q} \Psi_{z_j}^{*}(w)\|_{1/\alpha_1} \to 0 \quad \text{as} \quad q \to \infty.$$ 

Suppose that $1/\alpha_1 \leq 1$ (if $1/\alpha_1 > 1$ we do not take the $1/\alpha_1$ powers and, as before, use Minkowski’s inequality, the rest of the proof being the same). We have
\[
\left\| \psi'_*(w) - \frac{1}{q} \sum_{j=1}^{q} \psi'_*(w_j) \right\|_{1/\alpha_1} = \left\| e^{H(z)} C_{l}^{1-\beta(z)/1-\beta} F_{s}^{2} - \frac{1}{q} \sum_{j=1}^{q} e^{H(z)} C_{l}^{1-\beta(z)/1-\beta} F_{s_j}^{2} \right\|_{1/\alpha_1}
= \left\| e^{H(z)} C_{l}^{1-\beta(z)/1-\beta} \sum_{n,m} (C_{n}C_{m})^{\alpha(z)/\alpha} \epsilon_{n}\epsilon_{m} \Delta_{n}(w)\Delta_{m}(w) - \frac{1}{q} \sum_{j=1}^{q} e^{H(z)} C_{l}^{1-\beta(z)/1-\beta} \sum_{n,m} (C_{n}C_{m})^{\alpha(z)/\alpha} \epsilon_{n}\epsilon_{m} \Delta_{n}(w)\Delta_{m}(w) \right\|_{1/\alpha_1}
\leq \sum_{n,m} \left\{ \left\| e^{H(z)} C_{l}^{1-\beta(z)/1-\beta} (C_{n}C_{m})^{\alpha(z)/\alpha} \epsilon_{n}\epsilon_{m} - \frac{1}{q} \sum_{j=1}^{q} e^{H(z)} C_{l}^{1-\beta(z)/1-\beta} (C_{n}C_{m})^{\alpha(z)/\alpha} \epsilon_{n}\epsilon_{m} \right\| \Delta_{n}(w)\Delta_{m}(w) \right\|_{1/\alpha_1} \right\}^{1/\alpha_1}.
\]

But each of the terms
\[
e^{H(z)} C_{l}^{1-\beta(z)/1-\beta} (C_{n}C_{m})^{\alpha(z)/\alpha} - \frac{1}{q} \sum_{j=1}^{q} e^{H(z)} C_{l}^{1-\beta(z)/1-\beta} (C_{n}C_{m})^{\alpha(z)/\alpha}
\]
converge to 0 as \( q \to \infty \). This proves (10).

We thus have
\[
\int_{E_{i}} \left\| T \left( \psi'_* - \frac{1}{q} \sum_{j=1}^{q} \psi'_* \right) \right\| d\mu \leq \left\| T \left( \psi'_* - \frac{1}{q} \sum_{j=1}^{q} \psi'_* \right) \right\|_{1/\alpha_1} \{ \mu(E_{i}') \}^{1/1-\beta_{1}}
\leq M_{1} \left\| \psi'_* - \frac{1}{q} \sum_{j=1}^{q} \psi'_* \right\|_{1/\alpha_1} \{ \mu(E_{i}') \}^{1/1-\beta_{1}} \to 0 \text{ as } q \to \infty.
\]

A fortiori, we have
\[
e_{q} = \int_{E_{i}} \left\| T \psi'_* \right\| - \left\| T \left( \frac{1}{q} \sum_{j=1}^{q} \psi'_* \right) \right\| d\mu \to 0 \text{ as } q \to \infty.
\]

Thus, since
\[
\int_{E_{i}} \left\| T \psi'_* \right\| d\mu \leq e_{q} + \frac{1}{q} \sum_{j=1}^{q} \int_{E_{i}} \left\| T \psi'_* \right\| d\mu
\]
we obtain
\[
\psi'_*(z) \leq \lim_{\epsilon \to \infty} \frac{1}{q} \sum_{j=1}^{q} \psi'_*(z_{j}) = 1/2\pi \int_{0}^{2\pi} \psi'_*(z + \epsilon e^{it}) dt.
\]
We have thus proved that log Φ(z) is subharmonic. Thus our next task is to show that (6) implies the theorem.

Suppose that F is defined as in equation (1) where f is a simple function, but we no longer assume that ∥f∥₂/α = 1. We then have ∥f∥₂/α ≤ ∥F∥₂/α. Thus

\[ \| T(F^2) \|_{1/\beta}^\alpha = 1/\|F\|_{2/\alpha}^\alpha \cdot \| T(F^2) \|_{1/\beta} \leq 1/\|f\|_{2/\alpha}^\alpha \cdot \| T(F^2) \|_{1/\beta} \]

\[ = \| T(F^2) \|_{1/\beta} \leq K M_{1}^{(1-\alpha)} M_{2}^{\alpha} \]

since F/∥f∥₂/α is a function of type (1) for which this inequality was proved. But

\[ \| F^2 \|_{2/\alpha} = \| F^2 \|_{1/\alpha}. \]

We thus have, using the last part of (iii)

\[ \| T(F^2) \|_{1/\beta} \leq K M_{1}^{(1-\alpha)} M_{2}^{\alpha} \]

for any function of type (1) where f is simple.

Now let G be a function in H₁/α that is the square of a function F in H₂/α. Let f be a function in L₂/α(0, 2π) such that F(w) = 1/2πʃ₀⁽²π⁾((eⁱᵗ + w/eⁱᵗ - w))f(t)dt (i.e. f is the real part of F(eⁱᵗ)). Let \{fₙ\} be a sequence of simple functions such that ∥fₙ - f∥₂/α → 0 as n → ∞ and set Fₙ(w) = 1/2πʃ₀⁽²π⁾((eⁱᵗ + w)/(eⁱᵗ - w))fₙ(t)dt. Then F² - Fₙ² is in H₁/α ⊂ H₁/₂ (since 1/α ≥ 1/₂). Thus, in particular, F² is a sum of a function Fₙ in H₁/α₁ and a function F² - Fₙ² in H₁/α₂, showing, by (i), that T(F²) is defined. We also have

\[ | M T F^2 | - | T F^2 | \leq | T(F^2 - F_n^2) |. \]

But

\[ \| T(F^2 - F_n^2) \|_{1/\beta} \leq M_2 \| F^2 - F_n^2 \|_{1/\alpha} \leq M_2 \| F - F_n \|_{2/\alpha} \| F + F_n \|_{2/\alpha} \]

\[ \leq M_2 A_{1/\alpha}^2 \| f - f_n \|_{2/\alpha} \| f + f_n \|_{2/\alpha} \]

where C does not depend on the functions f and fₙ but is such that ∥g∥₂/α ≤ C₁∥g∥₂/α for any g in L₂/α(0, 2π) (C exists because the measure space is finite). The last factor is bounded since ∥fₙ - f∥₂/α → 0 and, by this very last fact, the entire right-hand side of the inequality goes to 0 as n → ∞. Thus there exists a subsequence \{ T(F_n^2) \} → 0 almost everywhere and, consequently, \{ T(F_n^2) \} → T(F²) almost everywhere, by (12). By Fatou's lemma, therefore, we have

\[ \int T(F^2) \|_{1/\beta} \, d\mu \leq \liminf \int T(F_n^2) \|_{1/\beta} \, d\mu \]

\[ \leq \liminf \{ K M_{1}^{(1-\alpha)} M_{2}^{\alpha} \} \]

\[ \{ K M_{1}^{(1-\alpha)} M_{2}^{\alpha} \} \]

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We have shown
\[ (13) \quad \| T(F^2) \|_{1/\beta} \leq KM_1^{(1-t_0)} M_2^{t_0} \| F \|_{1/\alpha} \]
for any function \( F \) in \( H_{2/\alpha} \).

But any function \( G \) in \( H_{1/\alpha} \) is a sum of two functions, \( G_1 \) and \( G_2 \), with no zeros, both in \( H_{1/\alpha} \), such that \( \| G_1 \|_{1/\alpha} = \| G \|_{1/\alpha} \) and \( \| G_2 \|_{1/\alpha} \leq 2 \| G \|_{1/\alpha} \). This is easily seen by the Blaschke product decomposition, \( G = BG_1 \) where \( B \) is the Blaschke product which satisfies \( |B(z)| < 1 \) if \( |z| < 1 \), and \( G_1 \) has no zeros. Thus, \( G = (B - 1)G_1 + G_1 \) is the desired decomposition with \( G_2 = (B - 1)G_1 \). The norm inequalities follow from the fact that \( \| G_1 \|_{1/\alpha} = \| G \|_{1/\alpha} \) and \( |B - 1| \leq 2 \) in the unit circle. Since \( G_1 \) and \( G_2 \) have no zeros they can be written as the squares of their square roots, and, clearly, these square roots are in \( H_{2/\alpha} \). Thus, using (iii)
\[ \| TG \|_{1/\beta} = \| T(G_1 + G_2) \|_{1/\beta} \leq \| TG_1 \|_{1/\beta} + \| TG_2 \|_{1/\beta} \]
\[ \leq KM_1^{(1-t_0)} M_2^{t_0} \{ \| G_1 \|_{1/\alpha} + \| G_2 \|_{1/\alpha} \} \]
\[ \leq KM_1^{(1-t_0)} M_2^{t_0} \{ \| G \|_{1/\alpha} + 2 \| G \|_{1/\alpha} \} = 3KM_1^{(1-t_0)} M_2^{t_0} \| G \|_{1/\alpha}. \]
This proves the theorem.

References


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