EIGENVALUES AND FUNCTIONAL EQUATIONS

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1. **Introduction.** The purpose of this paper is to illustrate how the techniques of the theory of dynamic programming, [1], may be used to convert a number of eigenvalue problems, where one is interested only in maximum or minimum values, into problems involving recurrence relations.

In turn we shall treat Jacobi matrices, some special types of quadratic forms possessing certain features of regularity, and finally Sturm-Liouville problems. The connection between Sturm-Liouville problems and dynamic programming has already been discussed in [2], using an approach different from that we shall present here.

The method discussed below is not only useful for computational purposes, but provides a method for studying the analytic dependence of the maximum and minimum eigenvalues upon the analytic structure of the matix.

2. **Jacobi matrices.**[1] Let us consider the Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & & 0 \\ a_1 & b_2 & a_2 & \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ 0 & a_{N-1} & b_{N-1} & a_{N-1} \\ & & a_{N-1} & b_N \end{pmatrix} \tag{1}$$

and the associated quadratic form

$$Q(x) = \sum_{k=1}^{N} b_k x_k^2 + 2 \sum_{k=1}^{N-1} a_k x_k x_{k+1}. \tag{2}$$

Define the sequence of functions

$$\begin{aligned} f_N(y) &= \operatorname*{Max}_{x_N^2=1} [b_N x_N + 2y x_N], \\ f_R(y) &= \operatorname*{Max}_{\{x\}} \left[\sum_{k=R}^{N} b_k x_k^2 + 2y x_R + 2 \sum_{k=R}^{N-1} a_k x_k x_{k+1} \right], \end{aligned} \tag{3}$$

where the maximization is over the region

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[1] Other computational techniques are treated in papers of W. Karush, [4] and C. Lanzcos, [5].

$$\sum_{k=R}^{N} x_k^2 = 1. \tag{4}$$

The maximum characteristic root of J is clearly $f_1(0)$.

Let us now show that we can obtain a recurrence relation connecting the members of the sequence $\{f_R(y)\}$. Write

$$\begin{aligned} f_R(y) = \operatorname*{Max}_{\{x\}} \Bigg[b_R x_R^2 + 2yx_R + \sum_{k=R+1}^{N} b_k x_k^2 + 2a_R x_R x_{R+1} \\ + 2 \sum_{k=R+1}^{N-1} a_k x_k x_{k+1} \Bigg]. \end{aligned} \tag{5}$$

Once x_R has been chosen, the problem of choosing the remaining x_k is quite similar to the original, with R transformed into $R+1$ and the constraint on the remaining x_k taking the form

$$\sum_{k=R+1}^{N} x_k^2 = 1 - x_R^2. \tag{6}$$

Let us then set

$$x_k = (1 - x_R^2)^{1/2} z_k, \qquad k = R+1, \cdots, N, \tag{7}$$

so that the constraint on z_k is $\sum_{k=R+1}^{N} z_k^2 = 1$.

We then have

$$\begin{aligned} f_R(y) = \operatorname*{Max}_{\{x\}} \Bigg[b_R x_R^2 + 2yx_R + (1 - x_R^2) \Bigg[\sum_{k=R+1}^{N} b_k z_k^2 + \frac{2a_R x_R z_{R+1}}{(1 - x_R^2)^{1/2}} \\ + 2 \sum_{k=R+1}^{N-1} a_k z_k z_{k+1} \Bigg] \Bigg]. \end{aligned} \tag{8}$$

Employing the "principle of optimality," [1], we obtain the recurrence relation

$$f_R(y) = \operatorname*{Max}_{x_R^2 \leq 1} [b_R x_R^2 + 2yx_R + (1 - x_R^2) f_{R+1}(a_R x_R/(1 - x_R^2)^{1/2})], \tag{9}$$

for $R = 1, 2, \cdots, N-1$.

A similar recurrence relation may be obtained for the minimum eigenvalue with Min replacing Max.

3. **Extensions.** Similar recurrence relations, of more complicated form, may be obtained from the consideration of matrices of the form

$$(1) \qquad J = \begin{pmatrix} b_1 & a_1 & c_1 & & 0 \\ a_1 & b_2 & a_2 & c_2 & \\ c_1 & a_2 & b_3 & a_3 & c_3 \\ & & & \ddots & \\ & 0 & & & \end{pmatrix}.$$

The basic sequence is

$$(2) \qquad f_R(u, v) = \operatorname{Max}\left[\sum_{k=R}^{N} b_k x_k^2 + 2 \sum_{k=R}^{N-1} a_k x_k x_{k+1} + 2 \sum_{k=R}^{N-2} c_k x_k x_{k+2} + 2u x_R + 2v x_{R+1} \right].$$

4. Some special classes of quadratic forms. Consider the following three special classes of quadratic forms

$$\text{(a) } Q_1 = (ax_1)^2 + (x_1 + ax_2)^2 + \cdots + (x_1 + x_2 + \cdots + x_{N-1} + ax_N)^2,$$

$$(1) \quad \text{(b) } Q_2 = x_1^2 + (x_1 + ax_2)^2 + \cdots + (x_1 + ax_2 + \cdots + a^{N-1}x_N)^2,$$

$$\text{(c) } Q_3 = x_1^2 + (x_1 + ax_2)^2 + (x_1 + ax_2 + (a+b)x_3)^2 + \cdots + (x_1 + ax_2 + (a+b)x_3 + \cdots + (a + (N-2)b)x_N)^2.$$

For the first quadratic form, define the sequence

$$(2) \quad f_R(y) = \operatorname*{Max}_{\{x\}} [(y + ax_R)^2 + (y + x_R + ax_{R+1})^2 + \cdots + (y + x_R + x_{R+1} + \cdots + x_{N-1} + ax_N)^2],$$

over the region $\sum_{k=R}^{N} x_R^2 = 1$.

As above, we obtain the recurrence relation

$$(3) \quad f_R(y) = \operatorname*{Max}_{x_r^2 \leq 1} [(y + ax_R)^2 + (1 - x_R^2) f_{R+1}((y + x_R)/(1 - x_R^2)^{1/2})].$$

Recurrence relations of similar form may be obtained for the other quadratic forms and for the minimum characteristic roots.

Many other special classes of quadratic forms can be constructed to yield simple recurrence relations. The form

$$(4) \qquad \sum_{k=1}^{N} [(x_k - a_k)^2 + b_k(x_k - x_{k-1})^2]$$

is discussed in [3].

5. **Eigenvalue problems.** If $\phi(x)$ is a continuous function over $[0, 1]$, uniformly positive so that $\phi(x) \geqq a^2 > 0$, the problem of determining the values of λ which yield nontrivial solutions of

$$\begin{aligned} u'' + \lambda\phi(x)u &= 0, \\ u(0) = u(1) &= 0, \end{aligned} \tag{1}$$

is equivalent to the problem of determining the relative minima of

$$J(u) = \int_0^1 u'^2 dx, \tag{2}$$

subject to the constraints

$$\begin{aligned} &\text{(a)} \int_0^1 \phi(x)u^2 dx = 1, \\ &\text{(b)}\ u(0) = u(1) = 0. \end{aligned} \tag{3}$$

We shall consider here only the absolute minimum. Using a different approach, a functional equation connected with this quantity was derived in [2]. Here we use the following approximate technique. Consider the problem of minimizing

$$J = \sum_{k=1}^{N} (u_k - u_{k-1})^2 \Delta, \tag{4}$$

subject to the restrictions

$$\begin{aligned} &(a)\ \sum_{k=1}^{N-1} \phi_k u_k^2 \Delta = 1, \\ &\text{(b)}\ u_0 = x, \qquad u_N = 0. \end{aligned} \tag{5}$$

Set $\phi_k = g_k^2$, $x_k = g_k u_k$, and absorb the Δ factor, obtaining the problem of minimizing

$$J(x) = \sum_{k=1}^{N} \left(\frac{x_k}{g_k} - \frac{x_{k-1}}{g_{k-1}} \right)^2, \tag{6}$$

subject to

$$\begin{aligned} &\text{(a)}\ \sum_{K=1}^{N-1} x_k^2 = 1, \\ &\text{(b)}\ x_0 = zg_0, \qquad x_N = 0. \end{aligned} \tag{7}$$

Define the sequence

$$f_R(z) = \operatorname{Min} \sum_{k=R}^{N} \left(\frac{x_k}{g_k} - \frac{x_{k-1}}{g_{k-1}} \right)^2, \tag{8}$$

with $x_{R-1} = zg_{R-1}$, over $\sum_{k=R}^{N} x_k^2 = 1$, $x_N = 0$.

We have

$$\begin{aligned} f_N(z) &= \operatorname*{Min}_{x_N^2=1} \left(\frac{x_N}{g_N} - z \right)^2 \\ &= \operatorname{Min} \left[\left(\frac{1}{g_N} - z \right)^2, \quad \left(\frac{1}{g_N} + z \right)^2 \right], \end{aligned} \tag{9}$$

and

$$f_R(z) = \operatorname*{Min}_{\{x\}} \left[\left(\frac{x_R}{g_R} - z \right)^2 + (1 - x_R^2) f_{R+1}(x_R/g_R(1 - x_R^2)^{1/2}) \right]. \tag{10}$$

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