MODULES WITHOUT INVARIANT BASIS NUMBER

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In a recent paper,[1] [p. 190] the author constructed a ring over which a finitely based module has invariant basis number if and only if it has a basis of length <2. It was indicated that this construction was generalizable, and this proceeds as follows:

Let R be a word ring (with unit) in symbols $\{a_{ij}, b_{st}\}$ ($j, s=1, \cdots, n\geqq 2; i, t=1, \cdots, n+1$) over the rationals. Let A and B be matrices whose elements are respectively $\{a_{ij}\}$ and $\{b_{st}\}$, and consider the relations of

$$(1) \qquad AB - I_{n+1} = 0, \qquad BA - I_n = 0.$$

It is clear that the method of proof [Lemma 1, p. 191] is applicable, and thus for each $\alpha\in R$ we may obtain in a finite number of steps a unique normal form $N(\alpha)$ not containing any of the leading words of the left-hand members of relations (1). If H is the two-sided ideal whose basis is the set of all elements of $AB-I_{n+1}$ and $BA-I_n$, we accordingly have an effective means of deciding whether or not $\alpha\in H$ (namely $\alpha\in H$ if and only if $N(\alpha)=0$). The quotient ring $K=R/H$ may also be regarded as a word ring in $\{a_{ij}, b_{st}\}$ all of whose members are reduced to normal form.

It may also be verified that K contains no zero divisors, the proof following that of [Lemma 2, p. 192]. Note that if the *degree* $d[\alpha]$ is defined to be the length of the longest word in α, this proof also shows that $d[\alpha\beta]=d[\alpha]+d[\beta]$. According to the remarks of [p. 193, footnote] it is clear that a module over K with a basis of length 1 has invariant basis number, while a module with a basis of length $\geqq n$ does not. This leaves the question open for a module over K with basis of length q $(1<q<n)$. It is the purpose of the present paper to show that such a module also has invariant basis number.

A word y is said to be *similar* to x if it differs from x only in either or both (a) the first subscript of its first symbol, (b) the second subscript of its last symbol; y is *left* {*right*} *similar* to x if only (a) {(b)} applies. Let $\{m_i\}$ $(i=1, \cdots, s)$ be a partition of the integer m. If x is a word of length m, a word y is said to be *compatible* with x (relative to the partition $\{m_i\}$) if $x=x_1x_2\cdots x_s$ and $y=y_1y_2\cdots y_s$, where $d[x_i]=d[y_i]=m_i$ with y_1 right similar to x_1, y_s left similar to x_s, and

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[1] William G. Leavitt, *Modules over word rings*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 188–193. References to this paper will be enclosed in brackets.

y_i similar to x_i $(1<i<s)$. Note that if u and v are words (in normal form) such that $d[u]=\sum_1^k m_i$ and $d[v]=\sum_{k+1}^s m_i$, then if the product uv is compatible with x before normalization, all its longest words remain compatible after normalization. An $\alpha\in K$ is *homogeneous* if its words are all of the same length.

Suppose $\{\alpha_i, \beta_i\}$ $(i=1, \cdots, q; 1<q<n)$ is a set of homogeneous members of K (in normal form) such that $d[\alpha_1]\geqq d[\alpha_2]\geqq \cdots \geqq d[\alpha_q]$ and such that for all i we have $d[\alpha_i\beta_i]=m$. Suppose further that all longest words in any of the products $\alpha_i\beta_i$ are compatible relative to the partition of m given by the nonzero numbers in the sequence

$$d[\alpha_q],\ d[\alpha_{q-1}]-d[\alpha_q],\ \cdots,\ d[\alpha_1]-d[\alpha_2],\ d[\beta_1]. \tag{2}$$

LEMMA 1. *If the above set $\{\alpha_i, \beta_i\}$ also satisfies the condition $d[\sum\alpha_i\beta_i]<m$, then (possibly relabeling those α_i for which $d[\alpha_i]=d[\alpha_1]$) there exist $\{\theta_i\}$ $(i=2, \cdots, q)$ such that $d[\alpha_1+\sum_2^q\alpha_i\theta_i]<d[\alpha_1]$.*

If any α_k is constant, the lemma follows trivially, for we may take $\theta_k=-\alpha_1/\alpha_k$, $\theta_i=0$ $(i\neq k)$. Similarly, if β_1 is constant we may choose $\theta_i=\beta_i/\beta_1$. Accordingly we suppose that all $d[\alpha_i]$, $d[\beta_i]>0$.

By compatibility the words of any particular α_r all end with either a_{pi} or b_{pi} (for fixed p), those of β_r all begin with either a_{jq} or b_{jq} (for fixed q). Call a product $\alpha_r\beta_r$ of type ab if the words of α_r end in a_{pi} while those of β_r begin with b_{jq}. A similar description applies to products of type ba, aa, or bb.

Let us suppose that the partition of m mentioned above has s members, then also by compatibility the words of α_1 have in general $2s-3$ variable indices, while those of β_1 have a single variable index. Call these indices $i, \cdots, j$ and k. If $d[\alpha_1]=\cdots=d[\alpha_t]>d[\alpha_{t+1}]\geqq\cdots$, this also applies to all α_r, β_r for which $1\leqq r\leqq t$. Let the coefficient of the word (of such an α_r) with variable indices $i, \cdots, j$ be $c^r_{i\cdots j}$, and of the word of β_r with index k be d^r_k. If $\alpha_r\beta_r$ is a product of type ab or ba, the coefficients of its longest words are $c^r_{i\cdots j}d^r_k-\delta_{jk}c^r_{i\cdots 1}d^r_1$ (where $\delta_{jk}=1$ if $j=k$, 0 otherwise), while if the product is of type aa or bb the second term is omitted. Proceeding to those α_r for which $d[\alpha_r]=d[\alpha_{t+1}]<d[\alpha_t]$, the words of such an α_r have $2s-5$ variable indices $i, \cdots$ and those of the corresponding β_r have three $\cdot jk$. If such an $\alpha_r\beta_r$ is a product of type ab or ba its longest words have coefficients $c^r_{i}\cdots d^r_{\cdot jk}-\delta_{\cdot\cdot}c^r_{i}\cdots_1 d^r_{1jk}$ where the subscripts of $\delta_{\cdot\cdot}$ are the last of $c^r_i\cdots$ and the first of $d^r_{\cdot jk}$. Again, if the product is of type aa or bb, the second term is omitted. Similar expressions are evidently obtained for the coefficients of the longest words of all $\alpha_i\beta_i$ $(i=1, \cdots, q)$.

Now $q<n$ and each variable index has range either 1 to n or 1 to

$n+1$, so each variable index has at least $q+1$ values. Also $d[\alpha_q]>0$, so at least one $c_i^q \neq 0$ (say, for definiteness, it is among $c_2^q, \cdots, c_{q+1}^q$). We now choose, for the moment, the subscripts $i, \cdots, j$ as follows: the second and succeeding subscripts will be fixed, with the second equal to 1, the last $q+1$, and no two adjacent subscripts equal. Since $d[\alpha_1\beta_1 + \cdots + \alpha_q\beta_q] < m$, it follows that all the coefficients of its longest words are zero. Thus, with the above choice of subscripts, we have

$$c^{1}_{i1\cdots q+1} d^{1}_{k} + \cdots + c^{q}_{i} d^{q}_{1\cdots q+1k} = 0, \tag{3}$$

where we choose $i=2, 3, \cdots, q+1$ and $k=1, \cdots, q$.

Since at least one of the $c_i^q \neq 0$, the rank of the coefficient matrix of (3) is at least one. Thus the set of vectors $(d_k^1, \cdots, d^q_{1\cdots q+1k})$ is dependent. We distinguish the following cases:[2]

Case I. $d_1^1 = \cdots = d_1^t = 0$. Since $d[\beta_1]>0$, at least one, say $d_u^1 \neq 0$. Then for any choice of the subscripts $i, \cdots, j$ we have

$$\begin{aligned} &c^{1}_{i\cdots j} d^{1}_{u} + \cdots + c^{t}_{i\cdots} d^{t}_{u} + c^{t+1}_{i\cdots} d^{t+1}_{\cdot ju} - \delta_{\cdot\cdot} c^{t+1}_{i\cdot\cdot 1} d^{t+1}_{1ju} + \cdots \\ &\qquad + c^{q}_{i} d^{q}_{\cdots ju} - \delta_{i\cdot} c^{q}_{1} d^{q}_{1\cdots ju} = 0. \end{aligned} \tag{4}$$

We choose $\theta_r = d_u^r/d_u^1$ for $(r \leqq t)$, while for $r>t$ we form the words of θ_r by dropping from the words of β_r the portions similar to β_1. If the variable subscripts of a resulting word are $\cdots j$, we choose for its coefficient $d^r{}_{\cdots ju}/d_u^1$ (that is, the coefficient of $c_i^r \cdots$ in the equation derived from (4) by dividing by d_u^1).

Case II. Some $d_1^h \neq 0$ (by relabelling, this becomes $d_1^1 \neq 0$) and for some u all $d_u^1 = \cdots = d_u^t = 0$. Then for any choice of subscripts $i, \cdots, j$ we have

$$\begin{aligned} &c^{1}_{i\cdots j} d^{1}_{1} + \cdots + c^{t}_{i\cdots j} d^{t}_{1} + \cdots + c^{q}_{i} d^{q}_{\cdots j1} - \delta_{i\cdot} c^{q}_{1} d^{q}_{1\cdots j1} = 0 \qquad (j \neq 1), \\ &-c^{1}_{i\cdots 1} d^{1}_{1} - \cdots - c^{t}_{i\cdots 1} d^{t}_{1} + \cdots + c^{q}_{i} d^{q}_{\cdots uu} - \delta_{i\cdot} c^{q}_{1} d^{q}_{1\cdots uu} = 0. \end{aligned} \tag{5}$$

Again $\theta_r = d_1^r/d_1^1$ $(r \leqq t)$, and for $r>t$ the coefficients of θ_r (with subscripts $\cdots j$) are $d^r{}_{\cdots j1}/d_1^1$ if $j \neq 1$ and $-d^r{}_{\cdots uu}/d_1^1$ if $j=1$.

Case III. For each k at least one of $d_k^1, \cdots, d_k^t \neq 0$. Since the vectors $(d_k^1, \cdots, d^q_{1\cdots q+1k})$ are dependent, there must be some u such that $(d_u^1, \cdots) = \sum_{k=1}^{u-1} h_k(d_k^1, \cdots)$ for some set of constants $\{h_k\}$.

[2] Remark that we are considering the worst situation, namely that in which all of the products $\alpha_i\beta_i$ are of type ab or ba. It is clear that a similar, though simpler, treatment may be used when some of the products are of type aa or bb. This is especially so if $\alpha_1\beta_1$ is of this type, for then we have essentially only the following Case I.

By relabelling, let $d_u^1 \neq 0$. Then

$$(6)\quad c^{1}_{i\cdots j}d^{1}_{u} + \cdots + c^{t}_{i\cdots j}d^{t}_{u} + \cdots + c^{q}_{i}d^{q}_{\cdots ju} - \delta_{i}.c^{q}_{1}d^{q}_{1\cdots ju} = 0 \qquad (j \neq u).$$

$$(7)\quad c_{i\cdots u}d^{1}_{k} + \cdots + c^{t}_{i\cdots u}d^{t}_{k} + \cdots + c^{q}_{i}d^{q}_{\cdots uk} - \delta_{i}.c^{q}_{1}d^{q}_{1\cdots uk} = 0 \qquad (k = 1, \cdots, u-1).$$

If the equations of (7) are multiplied respectively by h_k and summed, the result is

$$(8)\quad c^{1}_{i\cdots u}d^{1}_{u} + \cdots + c^{t}_{i\cdots u}d^{t}_{u} + \cdots + c^{q}_{i}e^{q}_{\cdots uu} - \delta_{i}.c^{q}_{1}e^{q}_{1\cdots uu} = 0,$$

where $e^{r}_{\cdots uu} = \sum_{k=1}^{u-1} h_k d^{r}_{\cdots uk}$. Once again $\theta_r = d^r_u/d^1_u$ $(r \leqq t)$ and the coefficients of the words of θ_r $(r > t)$ are $d^{r}_{\cdots ju}/d^1_u$ (if $j \neq u$) or $e^{r}_{\cdots uu}/d^1_u$ (if $j = u$).

It is clear from (4), (5), or (6) and (8) (depending on which of the three cases applies), the above choice of θ_r gives coefficients of the longest words of $\sum \alpha_r\theta_r$ which are just sufficient to cancel the coefficients $c^{1}_{i\cdots j}$ of the words of α_1.

We now define an *elementary transformation* to be either a permutation (relabeling) $\{\alpha_i, \beta_i\} \rightarrow \{\alpha_j, \beta_j\}$ or a transformation of type $\{\alpha_i, \beta_i\} \rightarrow \{\alpha_i', \beta_i'\}$ where for some r

$$(9)\quad \begin{aligned} \alpha_r' &= \alpha_r + \sum_{r+1}^{q} \alpha_k\theta_k, & \beta_i' &= \beta_i & (i \leqq r),\\ \alpha_i' &= \alpha_i \ (i \neq r), & \beta_i' &= \beta_i - \theta_i\beta_r & (i > r). \end{aligned}$$

LEMMA 2. *Let* $\{\alpha_i, \beta_i\}$ $(i = 1, \cdots, q < n)$ *be members of* K *arranged so that* $d[\alpha_i] \geqq d[\alpha_{i+1}]$. *If* $m = \max d[\alpha_i\beta_i]$ *and* $d[\sum\alpha_i\beta_i] < m$, *then there exist* $\{\alpha_i^*, \beta_i^*\}$, *reached by a finite sequence of elementary transformations, such that* $\sum\alpha_i^*\beta_i^* = \sum\alpha_i\beta_i$ *and* $\max d[\alpha_i^*\beta_i^*] < m$.

Since the case $q = 1$ is impossible, it is sufficient to show that from case q follows either a case $\leqq q-1$ or the lemma. If for any i we have $d[\alpha_i\beta_i] < m$, then we already have a case $\leqq q-1$. Thus suppose all $d[\alpha_i\beta_i] = m$. We again use the partition (2) determined by the $d[\alpha_i]$ and $d[\beta_i]$. Let x be a longest word of α_1 and y of β_1, and let $\bar{\alpha}_1$ be the part of α_1 containing all (and only) words compatible with x relative to this partition. Similarly, $\bar{\beta}_1$ is the part of β_1 compatible with y. In general, $\bar{\alpha}_i$ is the part of α_i compatible with x or a first portion of x, while $\bar{\beta}_i$ is the part of β_i compatible with uy, where u is either 1 or a last portion of x. Clearly, all longest words of $\bar{\alpha}_i\bar{\beta}_i$ are compatible with xy, and are not combinable with any other word obtained from any product $\alpha_i\beta_i$. Thus $d[\sum\bar{\alpha}_i\bar{\beta}_i]<m$, while $d[\bar{\alpha}_i\bar{\beta}_i]=m$ for all i, and so the set $\{\bar{\alpha}_i, \bar{\beta}_i\}$ satisfies the conditions of Lemma 1. Hence (possibly after a permutation) there exist $\{\theta_i\}$ such that $d[\bar{\alpha}_1+\sum_2^q \bar{\alpha}_i\theta_i]<d[\bar{\alpha}_1]$. Now let

$$\alpha_1' = \alpha_1 + \sum_2^q \alpha_i\theta_i, \qquad \beta_1' = \beta_1,$$
$$\alpha_i' = \alpha_i \ (i \neq 1), \qquad \beta_i' = \beta_i - \theta_i\beta_1 \qquad (i > 1),$$

then $\sum\alpha_i'\beta_i' = \sum\alpha_i\beta_i$, and all words compatible with x have been eliminated from α_1.

This means that $\alpha_1\beta_1$ (dropping primes henceforth, for convenience) now contains no word compatible with xz for any word z (with $d[z]=d[y]$). Thus if now $\sum_1^q \alpha_i\beta_i$ contains such a word, equating its coefficient to zero will involve nothing from $\alpha_1\beta_1$. We consider first the case $d[\alpha_2]=d[\alpha_1]$. If α_2 contains a word compatible with x, either $d[\alpha_2\beta_2]<m$, in which case the lemma is proved, or all words compatible with x may be eliminated from (possibly a permuted) α_2 in exactly the manner above. This process may clearly be extended to eliminate x from all words α_i $(i=1, \cdots, t)$ for which $d[\alpha_i]=d[\alpha_1]$.

Now suppose $d[\alpha_{t+1}]<d[\alpha_1]$ and $x=uv$, where $d[u]=d[\alpha_{t+1}]$. If α_{t+1} contains a word compatible with u, while β_{t+1} contains a word compatible with vz (for any z), equating to zero coefficients of words in $\sum\alpha_i\beta_i$ compatible with xz again involves nothing from $\alpha_i\beta_i$ $(i\leqq t)$. Thus, in a similar way, u may be eliminated from (a possibly permuted) α_{t+1}. Thus we have (or could obtain) a situation in which either α_{t+1} contains no word compatible with u, or β_{t+1} contains no word compatible with vz. Clearly this process may be continued until for each i, either α_i contains no word compatible with u_i, or β_i contains no word compatible with v_iz (where $x=u_iv_i$ and $d[u_i]=d[\alpha_i]$). Thus all words compatible with xz have been eliminated from all $\alpha_i\beta_i$.

We now proceed by induction. We suppose that for a given r for which $d[\alpha_{r+1}]<d[\alpha_r]$, for any word u such that $d[u]=d[\alpha_r]$ we are able to perform a series of elementary transformations such that no word of $\alpha_1, \alpha_2, \cdots, \alpha_r$ begins with a word compatible with u, and if $u=u_iv_i$ with $d[u_i]=d[\alpha_i]$ $(i>r)$ for each i, either α_i contains no word compatible with u_i or no word of β_i begins with a word compatible with v_i.

Now suppose $d[hv]=d[\alpha_r]$ and $d[h]=d[\alpha_{r+1}]$, and we carry out the elimination described in the induction hypothesis. If hw is another such word, we wish to show that such an eliminition relative to hw does not destroy that already accomplished relative to hv. The transformations are of the type $\alpha_j \to \alpha_j + \sum_{j+1}^{q} \alpha_i\theta_i$ and $\beta_i \to \beta_i - \theta_i\beta_j$ $(i > j)$, for some fixed j. If $j \leqq r$ we know that no α_i $(i \leqq r)$ begins with a word compatible with hv, so could not restore such a word to α_j. Also for $i > r$ either (I) α_i contains no word compatible with the first part of hv or (II) β_i contains no word whose first part is compatible with the final part of hv. But in forming the words of θ_i we take the words of $\bar{\beta}_i$ and lop off the part compatible with $\bar{\beta}_j$. Since the remaining part is at least as long as the final part of hv, it follows that if (II) applies to β_i then it also applies to θ_i. Thus in either case $\alpha_i\theta_i$ cannot restore a word whose first portion is compatible with hv. Also, for $i > r$, if β_i contains no word whose first part is compatible with the last of hv, this is clearly also true of $\beta_i - \theta_i\beta_j$.

Now when $j > r$, we are engaged in removing from α_j words compatible with h or a first portion of h. Thus no such word would be restored to α_j. Furthermore, for any $j > r$ for which such elimination is to be performed, the corresponding β_j cannot contain a word whose first portion is compatible with the end of hv. Thus if β_i has no such word, $\beta_i - \theta_i\beta_j$ cannot.

We can thus eliminate as described above, all words beginning with a word compatible with hw, for any w, from all α_i $(i \leqq r)$; that is, all words beginning with a word compatible with h. Also if α_{r+1} has a word compatible with h, then β_{r+1} has no longest word beginning with any w. This would mean $d[\alpha_{r+1}\beta_{r+1}] < m$ and the lemma would follow. Thus suppose α_{r+1} has no word compatible with h, and similarly for the remaining α_i for which $d[\alpha_i] = d[\alpha_{r+1}]$. Finally, if $h = h_i w_i$ with $d[h_i] = d[\alpha_i]$ (and $d[w_i] \neq 0$), either α_i has no word compatible with h_i or β_i has no word whose first part is compatible with $w_i w$ for any w. This establishes the induction.

By induction, then, for any g for which $d[g] = d[\alpha_q]$, we may eliminate all words beginning with a word compatible with g from α_i $(i = 1, \cdots, q)$. Clearly no further elimination can restore such a word to any α_i and hence we may eventually eliminate all longest words from some α_i. Then $d[\alpha_i\beta_i] < m$ and the lemma follows.

LEMMA 3. *If* $\sum_1^q \alpha_i\beta_i = 1$ *then there exists a finite sequence of elementary transformations to* $\{\alpha_i^+, \beta_i^+\}$ *such that* α_1^+ *is a constant and* $\alpha_i^+ = 0$ $(i \geqq 2)$.

By a permutation $\sum \alpha_i\beta_i$ may be placed in condition to apply Lemma 2, so there exists $\sum \alpha_i^*\beta_i^* = 1$ such that $\max d[\alpha_i^*\beta_i^*] < \max d[\alpha_i\beta_i]$. This may be continued as long as $\max d[\alpha_i\beta_i] > 0$ (dropping * for convenience). But suppose $\max d[\alpha_i\beta_i] \leqq 0$. Since $\sum \alpha_i\beta_i = 1$, at least one $\alpha_i\beta_i =$ constant. By permutation we get $\alpha_n = k$

and we can eliminate all other α_i. Another permutation gives $\alpha_1=k$, $\alpha_i=0$ (all $i\geqq 2$).

THEOREM. *For each $n>1$ there exists a ring without zero divisors over which a finitely based module has invariant basis number if and only if it has a basis of length $<n$.*

According to previous remarks we need consider only case $n>2$ and we need only prove that a module with basis of length $q<n$ has invariant basis number. This will be true if for any m by q and q by m $(m>q)$ matrices P and Q, the relation $PQ=I_m$ is impossible.

Suppose $PQ=I_m$. The first row of P and first column of Q satisfies Lemma 3. Clearly a nonsingular matrix T exists such that the first row of $P'=PT$ and first column $Q'=T^{-1}Q$ receive any desired elementary transformation. Since the first row of P' is $(k_1, 0 \cdots 0)$ and $P'Q'=I_m$, it follows that the first row of Q' is $(1/k_1, 0 \cdots 0)$. By a similar process, applied to the second row and second column of P' and Q', we reach P'', Q'' whose second rows are all zero except for the first two elements. But by this process we would reach $P^*Q^*=I_m$, where the last $m-q$ columns of Q^* are zero.

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