IDEALS IN A CERTAIN BANACH ALGEBRA

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1. **Introduction.** The purpose of this paper is to improve a result of Bertram Yood in his paper [2]. We are concerned with a complex commutative Banach algebra X and a compact Hausdorff space Ω. Let $C(\Omega, X)$ denote the set of all continuous functions defined over Ω with values in X. $C(\Omega, X)$ is a complex commutative B-algebra if we define addition, multiplication, and scalar multiplication in the natural "pointwise" manner and if $|f|_{C(\Omega,X)} = \sup_{a\in\Omega} |f(a)|_X$ for $f\in C(\Omega,X)$. Here, $|\ \ |_X$ denotes the norm in X. Yood proves, in Lemma 5.1 of his paper, that if X has a unit and if every maximal ideal in $C(\Omega, X)$ is of the form $\{f\in C(\Omega, X) \mid f(a_0)\in M_0\}$ with $a_0\in\Omega$ and M_0 a maximal ideal in X, then $\mathfrak{M}(C(\Omega, X))$, the space of maximal ideals in $C(\Omega, X)$ topologized in the Gelfand sense, is homeomorphic with $\Omega\times\mathfrak{M}(X)$, i.e., the topological product of Ω with the space of maximal ideals in X. We will show that all the maximal ideals in $C(\Omega, X)$ are necessarily of the above form and, further, Yood's result on $\mathfrak{M}(C(\Omega, X))$ is true even if X lacks a unit. If X lacks a unit element, then $C(\Omega, X)$ lacks a unit and $\mathfrak{M}(X)$, $\mathfrak{M}(C(\Omega, X))$ denote the spaces of all *regular* maximal ideals in X and $C(\Omega, X)$ respectively.

2. **Proof of theorem.** Before establishing two lemmas which we will employ in proving our theorem, we point out that the B-algebra X can be considered as contained as a subset of $C(\Omega, X)$. This is accomplished merely by identifying each $x\in X$ with the constant mapping $\Omega\rightarrow x$. The elements of X will thus be viewed as constants in $C(\Omega, X)$.

LEMMA 1. *Let $f\in C(\Omega, X)$ and $\epsilon>0$. Then there exist $x_i\in X$ and continuous complex-valued functions f_i defined over Ω $(i=1, 2, \cdots, n)$ such that* $|f-\sum_{i=1}^n x_if_i|_{C(\Omega,X)}\leqq\epsilon$.[1]

PROOF. Let $S=f(\Omega)$. S is a compact set in X since Ω is compact and f is continuous. Take spheres of radius ϵ about each point in S. This is an open covering of S and, since S is compact, there is a finite open

Received by the editors April 7, 1956 and, in revised form, May 29, 1956.

[1] As the proof will show, this lemma is true if X is only a Banach *space*. The author is grateful to Dr. G. W. Booth for supplying the proof of the lemma and for pointing out that the method of a partition of unity subordinate to a covering of a compact set in a B-space is contained in the New York University lecture notes of L. Bers *Introduction to topology.*

subcovering $S(x_i, \epsilon)$ of S $(i=1, 2, \cdots, n)$, $x_i \in S$. We may find a partition of unity subordinate to this covering, i.e., we may find continuous functions on S, $\lambda_i(x)$ $(i=1, 2, \cdots, n)$, such that $0 \leqq \lambda_i(x) \leqq 1$, $\lambda_i(x)=0$ on $S-S(x_i, \epsilon)$ and $\sum_{i=1}^n \lambda_i(x) \equiv 1$ for $x \in S$. To see this, define $\mu_i(x) = \text{dist}\,(x, S-S(x_i, \epsilon))$ for $i=1, 2, \cdots, n$ where dist $(x, S-S(x_i, \epsilon))$ denotes the distance from $x \in S$ to the set $S-S(x_i, \epsilon)$. Each $\mu_i(x)$ is continuous and $\geqq 0$. We see that $\mu_i(x)=0$ if $x \in S-S(x_i, \epsilon)$, $\mu_i(x)>0$ if $x \in S(x_i, \epsilon)$. Write $\mu(x)=\sum_{i=1}^n \mu_i(x)$. Clearly $\mu(x) \neq 0$ for all $x \in S$ since $\{S(x_i, \epsilon)\}$ is a covering of S. Define $\lambda_i(x)=\mu_i(x)/\mu(x)$ for $x \in S$. These functions make up the desired partition of unity.

Let $\mathcal{C}(x_1, x_2, \cdots, x_n)$ denote the convex hull of the x_i's, i.e., $y \in \mathcal{C}(x_1, x_2, \cdots, x_n)$ if and only if there exist numbers $r_1, r_2, \cdots, r_n \geqq 0$ such that $\sum_{i=1}^n r_i = 1$ and $y = \sum_{i=1}^n r_i x_i$. Using the $\lambda_i(x)$ we may define $T: S \rightarrow \mathcal{C}(x_1, x_2, \cdots, x_n)$ as follows: $Tx = \sum_{i=1}^n \lambda_i(x) x_i$ $(x \in S)$. T is continuous and, further, $|Tx - x| \leqq \epsilon$ for all $x \in S$. For, $|Tx-x| = |\sum_{i=1}^n \lambda_i(x)x_i - x| = |\sum_{i=1}^n \lambda_i(x)x_i - (\sum_{i=1}^n \lambda_i(x))x| = |\sum_{i=1}^n \lambda_i(x) \cdot (x_i - x)| \leqq \sum_{i=1}^n \lambda_i(x)|x_i - x| \leqq \epsilon$. To complete the proof, define $f_i = \lambda_i(f)$. Then, $|Tf(a) - f(a)| = |\sum_{i=1}^n x_i \lambda_i(f(a)) - f(a)| \leqq \epsilon$ for all $a \in \Omega$. q.e.d.

LEMMA 2. *Every, not identically zero, continuous multiplicative linear functional in the B-algebra $C(\Omega, X)$ is of the form $\phi f = \phi_M f(a)$ for some $a \in \Omega$, $M \in \mathfrak{M}(X)$. Here, ϕ_M denotes the Gelfand homomorphism from X onto the complex numbers associated with M.*

PROOF. Suppose, firstly, that X has a unit e and that $|e|=1$ (if $|e|$ were not 1 we could renorm X so as to achieve this). Then ϕ is a nonzero continuous multiplicative linear functional on $eC(\Omega) = \{ef \in C(\Omega, X) \mid f \in C(\Omega)\}$ (here, $C(\Omega)$ denotes the B-algebra of continuous complex-valued functions defined over Ω). Also ϕ is a nonzero multiplicative linear functional on $X \subset C(\Omega, X)$. The last two statements concerning ϕ can be proved as follows: If ϕ were identically zero on $eC(\Omega)$ or X, then $\phi(xf) = \phi(ef) \cdot \phi(x) = 0$ for all $f \in C(\Omega)$ and all $x \in X$. Since linear combinations of functions xf with $f \in C(\Omega)$, $x \in X$ are dense in $C(\Omega, X)$ by Lemma 1, this would mean ϕ is identically zero in $C(\Omega, X)$ contrary to assumption.

It is easy to see that $eC(\Omega)$ is isometrically isomorphic with $C(\Omega)$. Hence, as is well known, there exists an $a \in \Omega$ such that $\phi(ef) = f(a)$ with $f \in C(\Omega)$. Also there exists an $M \in \mathfrak{M}(X)$ such that $\phi(x) = \phi_M(x)$ for all $x \in X \subset C(\Omega, X)$.

Suppose, now, that f is an arbitrary function in $C(\Omega, X)$. By Lemma 1, there exists a sequence $f_n \in C(\Omega, X)$ such that $f_n \rightarrow f$ in $C(\Omega, X)$-norm and furthermore $\phi(f_n)=\phi_M f_n(a)$. Since ϕ is continuous we have $\phi f=\phi_M f(a)$.

Suppose that X lacks an e. Then we imbed X, isometrically and isomorphically, in a Banach algebra X' with unit e in such a way that the maximal ideals of X' are the regular maximal ideals of X and X itself. This gives rise to an additional homomorphism of X' onto the complex numbers, namely ϕ_X, where $\phi_X(x)=0$ if $x\in X$ and $\phi_X(\lambda e)=\lambda$ for all complex numbers λ. The space $\mathfrak{M}(X')$ is the one-point-compactification of $\mathfrak{M}(X)$ by ϕ_X. By what we have already proved, the nonzero multiplicative functionals in $C(\Omega, X')$ are of the form $\phi_M f(a)$ and the additional functionals $\phi_X f(a)$. These latter functionals are all identically zero on $C(\Omega, X)$ so that the most general nonzero multiplicative functionals on $C(\Omega, X)$ are of the form $\phi_M f(a)$ with $M\in\mathfrak{M}(X)$, $a\in\Omega$. q.e.d.

COROLLARY. *The only regular maximal ideals in* $C(\Omega, X)$ *are of the form* $\{f\in C(\Omega, X) \mid f(a)\in M\}$ *with* $a\in\Omega$, $M\in\mathfrak{M}(X)$.

THEOREM. $\mathfrak{M}(C(\Omega, X))$ *is homeomorphic with* $\Omega\times\mathfrak{M}(X)$.

PROOF. By the corollary, above, there is a 1-1 correspondence between the points of $\mathfrak{M}(C(\Omega, X))$ and those of $\Omega\times\mathfrak{M}(X)$. Now, the topology in $\mathfrak{M}(C(\Omega, X))$ is that induced by the family $\mathcal{G}$ which consists of the functions $g_f(f\in C(\Omega, X))$ defined on $\Omega\times\mathfrak{M}(X)$ by $g_f(a, M)=\phi_M f(a)$. We must show that this topology is identical with the product topology. For this purpose we use the following result (see [1, p. 12]):

If $\mathfrak{F}$ is a family of continuous complex-valued functions vanishing at infinity on a locally compact space Π, separating the points of Π and not all vanishing at any point of Π, then the topology induced on Π by $\mathfrak{F}$ is identical with the given topology of Π.

We take $\Pi=\Omega\times\mathfrak{M}(X)$ and define $\mathfrak{F}$ as follows. For each positive integer n and each choice of $f_1, \cdots, f_n\in C(\Omega)$, $x_1, \cdots, x_n\in X$, there is a function h defined on $\Omega\times\mathfrak{M}(X)$ by $h(a, M)=\sum_{i=1}^{n}\phi_M(x_i f_i(a))$. Let $\mathfrak{F}$ be the family of all functions h so defined. Π is locally compact since Ω is compact and $\mathfrak{M}(X)$ is locally compact. Further, every function in $\mathfrak{F}$ is continuous in (a, M) over $\Omega\times\mathfrak{M}(X)$ since this is true for the functions $\phi_M f(a)$ with $f\in C(\Omega, X)$ (see [2, Lemma 2.2]).

Not all functions in $\mathfrak{F}$ vanish at any point $(a_0, M_0)\in\Omega\times\mathfrak{M}(X)$. For, let $x\in X$ be such that $x\notin M_0$ and pick $f\in C(\Omega)$ such that $f(a_0)\neq 0$. Then $\phi_{M_0}(xf(a_0))\neq 0$.

The functions in $\mathfrak{F}$ separate points in $\Omega\times\mathfrak{M}(X)$. Suppose $(a, M)\neq(b, N)$ with $a\neq b$. If $M=N$, take $x\notin M$ and $f\in C(\Omega)$ such that

$f(a)\neq f(b)$. Then $\phi_M(xf(a))\neq\phi_N(xf(b))$. If $M\neq N$, find an $x\in M$, $x\notin N$ and $f\in C(\Omega)$ such that $f(a)=0$ and $f(b)\neq 0$. Then $\phi_M(xf(a))\neq\phi_N(xf(b))$. If $a=b$ so that $M\neq N$ we may find $x\in M$, $x\notin N$. Choose $f\in C(\Omega)$ such that $f(a)=f(b)\neq 0$; then $\phi_M(xf(a))\neq\phi_N(xf(b))$.

Finally we show that functions in $\mathfrak{F}$ vanish at ∞ in $\Omega\times\mathfrak{M}(X)$. Suppose $\epsilon>0$ is given. If $\sum_{i=1}^n \phi_M(x_if_i(a))\in\mathfrak{F}$, then $\left|\sum_{i=1}^n f_i(a)\phi_M(x_i)\right| \leqq\epsilon$ if $(a, M)\notin\Omega\times(\bigcup_{i=1}^n \mathfrak{C}_i)$ where $|\phi_M(x_i)|<\delta$ if $M\notin\mathfrak{C}_i$, $\mathfrak{C}_i$ being compact sets in $\mathfrak{M}(X)(i=1, 2, \cdots, n)$, and where $\delta<\epsilon/nK$ with $K=\sup_{1\leqq i\leqq n}\sup_{a\in\Omega}|f_i(a)|$. The sets $\mathfrak{C}_i$ exist because each $\phi_M(x_i)$ vanishes at infinity in $\mathfrak{M}(X)$. Further, $\Omega\times(\bigcup_{i=1}^n \mathfrak{C}_i)$ is compact in $\Omega\times\mathfrak{M}(X)$ so that each function in $\mathfrak{F}$ vanishes at ∞ in $\Omega\times\mathfrak{M}(X)$.

Using the result quoted in the beginning of the proof we have shown that the product topology of $\Omega\times\mathfrak{M}(X)$ is identical with that induced by our family $\mathfrak{F}$. Since $\mathfrak{F}$ is smaller than the family $\mathfrak{J}$ we see that the topology induced by $\mathfrak{J}$ on $\Omega\times\mathfrak{M}(X)$ is stronger or equal to the topology induced by $\mathfrak{F}$. Since the topology of $\mathfrak{M}(C(\Omega, X))$ is precisely that induced on $\Omega\times\mathfrak{M}(X)$ by the family $\mathfrak{J}$, the proof will be completed by showing that the $\mathfrak{F}$- and $\mathfrak{J}$-topologies on $\Omega\times\mathfrak{M}(X)$ are identical. The family $\mathfrak{F}$ is contained in and is dense in $\mathfrak{J}$ in the uniform norm for continuous functions. For, suppose $f\in C(\Omega, X)$. Then, by Lemma 1, we can find $\sum_{i=1}^n x_if_i$, $f_i\in C(\Omega)$, $x_i\in X$ such that $\left|f-\sum_{i=1}^n x_if_i\right|_{C(\Omega,X)}\leqq\epsilon$. This means $\sup\left|\phi_Mf(a)-\sum_{i=1}^n\phi_M(x_if_i(a))\right| \leqq\epsilon$, where the sup is taken over all $(a, M)\in\Omega\times\mathfrak{M}(X)$. Hence $\mathfrak{F}$ is dense in $\mathfrak{J}$ in the uniform norm and it follows that the $\mathfrak{F}$- and $\mathfrak{J}$-topologies are identical. q.e.d.

3. **Concluding remarks.** The essential ideas in this paper are direct outgrowths of the author's Yale doctoral thesis *Group algebras of vector-valued functions*. Just as in the thesis we can show, using the characterization of $\mathfrak{M}(C(\Omega, X))$ as $\Omega\times\mathfrak{M}(X)$, that $C(\Omega, X)$ is regular if and only if X is regular, and that every proper closed ideal in $C(\Omega, X)$ is contained in a regular maximal ideal if X satisfies certain additional conditions. Further, results on kernels and hulls could be obtained in $C(\Omega, X)$. All these can be proved by minor and obvious changes in the proofs of the thesis.

References

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2. B. Yood, *Banach algebras of continuous functions*, Amer. J. Math. vol. 73 (1951) pp. 30–42.

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