SOME BANACH ALGEBRAS

EARL A. CODDINGTON

We define a class of commutative regular Banach algebras which contain simple examples of such algebras which are not self-adjoint. Conditions are given for the validity of the general Wiener theorem, and the denseness of certain translates. We use the Banach algebra terminology and results as presented in [**1**].

1. **The Banach algebra** A. Let $L^2(S)$ be the set of all complex-valued measurable functions, on a measure space S, whose magnitudes are square summable. We consider $L^2(S)$ as a Hilbert space, and assume the existence of a countable complete orthonormal set $\{\phi_k\}$ in $L^2(S)$, which has the additional property that $\phi_k \in L^\infty(S) \cap L^1(S)$ for all k. For the given $\{\phi_k\}$ let $\{\nu_k\}$ be any sequence of nonzero complex numbers satisfying the condition

$$\sum \|\phi_k\|_\infty^2 \|\phi_k\|_1 |\nu_k| \leqq 1, \tag{1}$$

where $\|\phi_k\|_\infty$, $\|\phi_k\|_1$ denote the norms of ϕ_k in $L^\infty(S)$ and $L^1(S)$ respectively. If $f, g \in L^1(S)$ we define the product $f * g$ by[1]

$$f * g = \sum (f, \phi_k)(g, \phi_k)\nu_k^c \phi_k.$$

Here $(f, \phi_k) = \int_S f\phi_k^c$, and in general $(f, \alpha) = \int_S f\alpha^c$ in case $f \in L^1(S)$, $\alpha \in L^\infty(S)$, or in case $f, \alpha \in L^2(S)$. By (1), the series converges in $L^1(S)$, and indeed

$$\|f * g\|_1 \leqq \|f\|_1 \|g\|_1 \qquad (f, g \in L^1(S)). \tag{2}$$

We have the following result:

$L^1(S)$ with $*$ as multiplication is a commutative Banach algebra A which is regular.

Proof. From (2) it follows that A is closed under multiplication and satisfies the norm requirement for the product. The definition of the product clearly shows that the commutative and associative laws for multiplication are valid. Thus A is a commutative Banach algebra.

It remains to prove that A is regular, and for this we determine the regular maximal ideal space $\mathfrak{M}$ for A. The points M of $\mathfrak{M}$ are in a one-to-one correspondence with the nonzero continuous homomorphisms of A onto the complex numbers. Let l_M be such a homo-

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[1] For any a, a^c denotes the complex conjugate of a.

morphism corresponding to $M \in \mathfrak{M}$. Then $l_M(\phi_k)$ cannot be zero for all k, for if the contrary is true, we have for any $f, g \in L^1(S)$,

$$l_M(f)l_M(g) = l_M(f * g) = \sum (f, \phi_k)(g, \phi_k)\nu_k^c l_M(\phi_k) = 0.$$

Since l_M can not be identically zero, this gives a contradiction. Let k be such that $l_M(\phi_k) \neq 0$. Then $l_M(f)l_M(\phi_k) = l_M(f * \phi_k) = (f, \phi_k)\nu_k^c l_M(\phi_k)$, or $l_M(f) = (f, \nu_k\phi_k)$. There is exactly one k such that $l_M(\phi_k) \neq 0$, for $l_M(\phi_j) = (\phi_j, \nu_k\phi_k) = 0$, if $j \neq k$. Thus corresponding to $M \in \mathfrak{M}$ there is a unique integer k such that $l_M(f) = (f, \nu_k\phi_k)$. Conversely every $\nu_k\phi_k$ generates in this way a continuous homomorphism onto the complex numbers, and the $f \in A$ satisfying $(f, \nu_k\phi_k) = 0$ form a regular maximal ideal M. Therefore $\mathfrak{M}$ can be identified with the integers $M \leftrightarrow k$.

For any $f \in A$ let $\hat{f}$ be the complex-valued function defined on $\mathfrak{M}$ by $\hat{f}(M) = \hat{f}(k) = (f, \nu_k\phi_k)$. Then A is said to be regular if, given any closed set C in $\mathfrak{M}$ and point M_0 not in C, there exists an $f \in A$ such that $\hat{f}(M) = 0$ on C, $\hat{f}(M_0) \neq 0$. For our A let C be any set in $\mathfrak{M}$ and M_0 a point not in C determined by the integer k. Then $\phi_k \in A$ has the property that $\widehat{\phi_k}(j) = \delta_{jk}\nu_j^c$, thus showing that A is regular.

The algebra A is semi-simple (the map $f \to \hat{f}$ is one-to-one) if and only if $(f, \phi_k) = 0$ for all k implies $f = 0$ almost everywhere. Suppose A is semi-simple. If $f \in A$, and $\hat{f}$ vanishes outside a compact set, then $(f, \phi_k) = 0$ for $|k| > N$, for some positive integer N. Thus

$$f = \sum_{|k| \leq N} (f, \phi_k)\phi_k,$$

since $(f - \sum_{|k| \leq N} (f, \phi_k)\phi_k, \phi_j) = 0$ for all j. It follows that the set of elements $f \in A$, such that $\hat{f}$ vanishes outside a compact set, is dense in A if and only if the sequence $\{\phi_k\}$ is dense in A. The general Wiener Tauberian theorem (in the form of the Corollary, p. 85, [1]) then takes the following form.

Suppose the sequence $\{\phi_k\}$, which defines the algebra A, satisfies the two conditions:

(a) $(f, \phi_k) = 0$, *for* $f \in L^1(S)$ *and all* k, *implies* $f = 0$ *almost everywhere*,

(b) $\{\phi_k\}$ *is dense in* $L^1(S)$.

Then every proper closed ideal is included in a regular maximal ideal.

2. **Some examples.** A commutative Banach algebra A is said to be self-adjoint if for every $f \in A$ there exists a $g \in A$ such that $\hat{g} = \hat{f}^c$. We give an example of a choice of $\{\phi_k\}$ and $\{\nu_k\}$ which lead to an algebra which is regular, but not self-adjoint. Let S be the real interval $-\pi \leqq x \leqq \pi$, and $\phi_k = (2\pi)^{-1/2}e^{-ikx}$, $k = 0, \pm 1, \pm 2, \cdots$. Clearly $\phi_k \in L^\infty(S) \cap L^1(S) \cap L^2(S)$. The condition for self-adjointness is that for each $f \in L^1(S)$ there exists a $g \in L^1(S)$ such that $\hat{g}(k) = (g, \nu_k\phi_k)$

$=(f, \nu_k\phi_k)^c$. If $\nu_k = |\nu_k| e^{i\theta_k}$, where $0 \leqq \theta_k < 2\pi$, this condition becomes $(g, \phi_k) = e^{2i\theta_k}(f, \phi_k)^c$. Let $f^-(x) = [f(-x)]^c$. Clearly $f^- \in L^1(S)$ if and only if $f \in L^1(S)$, and $(f, \phi_k)^c = (f^-, \phi_k)$. Thus $L^1(S)$ is self-adjoint if and only if, given any $f^- \in L^1(S)$, there exists a $g \in L^1(S)$ such that

$$(g, \phi_k) = e^{2i\theta_k}(f^-, \phi_k) \qquad (k = 0, \pm 1, \cdots).$$

Now the function f^- defined by

$$f^-(x) = \sum_{k=1}^{\infty} k^{-1/4} e^{ikx}$$

is in $L^1(-\pi, \pi)$, but a sequence of $\pm$ signs exist for which

$$\sum_{k=1}^{\infty} \pm\ k^{-1/4} e^{ikx}$$

is not the Fourier series of any function in $L^1(-\pi, \pi)$ (see [**2**, p. 212]). If we choose the θ_k so that $e^{2i\theta_k}$ gives such a sequence of signs, we see that A will not be self-adjoint. Note that this A is semi-simple and satisfies the conditions (a) and (b) for the general Wiener theorem.

Further examples are obtained by letting the ϕ_k be the orthonormal eigenfunctions of a self-adjoint ordinary differential operator on a finite closed interval $S: a \leqq x \leqq b$. The ϕ_k are continuous, and it is known that both (a) and (b) are valid.

Certain singular self-adjoint differential operators have a pure point spectrum. For example, the problem $-u'' + q(x)u = \lambda u$, on $S: -\infty < x < \infty$, is self-adjoint and has a pure point spectrum if q is real-valued, continuous, and $q(x) \to \infty$ as $x \to \pm \infty$. It is known that in this case the orthonormal eigenfunctions ϕ_k are in $L^\infty(S) \cap L^1(S) \cap L^2(S)$. For the particular case where $q(x) = x^2$ the ϕ_k are the Hermite functions, and the conditions (a) and (b) are valid.

3. **The denseness of certain translates.** For each $y \in S$ let T_y be defined by

$$T_y f(x) = \sum (f, \phi_k)\nu_k^c [\phi_k(y)]^c \phi_k. \tag{3}$$

From (1) it follows that $\|T_y f\|_1 \leqq \|f\|_1$ for every $f \in L^1(S)$, and an equivalent definition of $f * g$ is

$$f * g(x) = \int_S T_y f(x) g(y) dy.$$

We call $T_y f$ the translate of f by y. Let $\mathfrak{T}_f$, for a given $f \in L^1(S)$, be the set of all finite linear combinations, with complex coefficients, of translates of f. It does not appear obvious that the closure of $\mathfrak{T}_f$ is an ideal in A, in case A satisfies (a) and (b). Nevertheless it is easy to show directly the following result:

Suppose $\{\phi_k\}$ *is dense in* A, *and* $f \in A$ *is such that* $(f, \phi_k) \neq 0$ *for any* k. *Then* $\mathfrak{T}_f$ *is dense in* A.

PROOF. By the Hahn-Banach theorem $\mathfrak{T}_f$ is dense in A if and only if the only bounded linear functional l which satisfies $l(g) = 0$ for all $g \in \mathfrak{T}_f$ is the identically zero one. Suppose $l(g) = 0$ for all $g \in \mathfrak{T}_f$. In particular $l(T_y f) = 0$ for all $y \in S$. Thus, using the definition (3),

$$\sum (f, \phi_k) \overset{c}{\nu_k} [\phi_k(y)]^c l(\phi_k) = 0 \qquad (y \in S),$$

where the series converges in $L^\infty(S)$ by (1). Since $\phi_j \in L^1(S)$ we have $0 = (\phi_j, 0) = (f, \phi_j)\nu_j^c l(\phi_j)$, and, since $(f, \phi_j)\nu_j^c \neq 0$ for any j, this implies $l(\phi_j) = 0$ for all j. This in turn implies that l is the identically zero linear functional, since the ϕ_j are dense in A.

REFERENCES

1. L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953.
2. A. Zygmund, *Trigonometrical series*, Warsaw, 1935.

MATHEMATICS INSTITUTE, UNIVERSITY OF COPENHAGEN AND
UNIVERSITY OF CALIFORNIA, LOS ANGELES