# INDICATRIC TORSION IN A SUBSPACE OF A RIEMANNIAN SPACE

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1. **Definition.** In a former paper [4] the author defined indicatric torsion of a vector field in a direction to generalize the concept of geodesic torsion of a curve in a surface of a Euclidean three dimensional space. The present note extends the investigation to a subspace of a Riemannian space. Special concern is for hypersurfaces in a Riemannian space and in a Euclidean space.

Let $V_n$ be a Riemannian space with fundamental quadratic form

$$\phi = g_{ij}dx^i dx^j \qquad (i, j = 1, \cdots, n)$$

which is immersed in a Riemannian space $V_m$ with fundamental quadratic form

$$\psi = a_{\alpha\beta}dy^\alpha dy^\beta \qquad (\alpha, \beta = 1, \cdots, m).$$

Let $V_n$ be defined in $V_m$ by equations of the form

$$y^\alpha = y^\alpha(x^1, \cdots, x^n)$$

where the functional matrix $\|\partial y/\partial x\|$ is of rank $n$.

Let $v: v^\alpha = y^\alpha{}_{,i} v^i$ be a unit vector field in $V_n$ and let $C: x^i = x^i(s)$ be a curve in $V_n$, $s$ being its arc length. We assume $v$ and $C$ to be of class not less than 2 and 3, respectively. Let $N_\sigma|^\alpha$ for $\sigma = n+1, \cdots, m$ denote $m-n$ mutually orthogonal unit vectors normal to $V_n$. Define

$$\eta_1|^\alpha = v^\alpha, \qquad \eta_2|^\alpha = {}_v k_1 N_\nu|^\alpha, \tag{1.1}$$
$$\eta_{r+1}|^\alpha = \eta_r|^\alpha{}_{;i}\frac{dx^i}{ds} \qquad (r = 2, \cdots, m-1)$$

where

$${}_v k_1 = a_{\alpha\beta} N_\nu|^\alpha \left(\frac{\partial^2 y^\beta}{\partial x^i \partial x^j} + \frac{\partial y^\gamma}{\partial x^i}\frac{\partial y^\delta}{\partial x^j}\Gamma^\beta{}_{\gamma\delta} - \frac{\partial y^\beta}{\partial x^k}\left\{\begin{matrix} k \\ i \quad j\end{matrix}\right\}\right) v^i \frac{dx^j}{ds},$$

$\Gamma^\beta{}_{\gamma\delta}$ = Christoffel symbols in $V_m$,

$$\left\{\begin{matrix} k \\ i \quad j\end{matrix}\right\} = \text{Christoffel symbols in } V_n.$$

If the vectors $\eta_\delta|^\alpha$ for $\delta = 1, \cdots, m$ are assumed to be linearly inde-

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pendent, the following linear combinations of them for $p=1, \cdots m$, form a set of $m$ mutually orthogonal vectors [1, pp. 103–104]:

$$\lambda_p|^\alpha = \left(\frac{b_p}{b_{p-1}}\right)^{1/2} \eta_\lambda|^\alpha B^\lambda{}_p \qquad (\lambda, \mu = 1, \cdots, p) \tag{1.2}$$

where

$$b_0 = 1, \qquad b_p = |b^\mu{}_\lambda|,$$
$$b^\lambda{}_\mu = b^\mu{}_\lambda = a_{\alpha\beta}\eta_\lambda|^\alpha \eta_\mu|^\beta,$$
$$b^\mu{}_\lambda B^\lambda{}_\rho = \delta^\mu{}_\rho.$$

The formulas of Frenet for $v$ along $C$ in $V_m$ read as follows:

$$\lambda_p|^\alpha{}_{;i} \frac{dx^i}{ds} = -{}_vK_{p-1}\lambda_{p-1}|^\alpha + {}_vK_p\lambda_{p+1}|^\alpha \tag{1.3}$$

where ${}_vK_p$ for $p=1, \cdots, m-1$ are, respectively, the associate curvatures of order $1, \cdots, m-1$ of the vector field $v$ for the curve $C$.

From (1.2) we have

$$\lambda_2|^\alpha = N_\nu|^\alpha,$$

and from (1.3) we have for $p=2$

$$N_\nu|^\alpha{}_{;i} \frac{dx^i}{ds} = -{}_vK_1 v^\alpha + {}_v\tau_g \lambda_3|^\alpha \tag{1.4}$$

where ${}_v\tau_g$ is employed to denote ${}_vK_2$ in the case under consideration.

The scalar, ${}_v\tau_g$, is called the *indicatric torsion of $v$ along $C$ in $V_n$ in $V_m$ relative to $N_\nu|^\alpha$*. When $V_n$ is a surface in a Euclidean three dimensional space, ${}_v\tau_g$ is the indicatric torsion of $v$ in the direction of $C$ as introduced in [4]. If $v$ is furthermore tangent to $C$, ${}_v\tau_g$ becomes the geodesic torsion of $C$ as defined in classical differential geometry.

2. **Formulas.** Denote the unit vector tangent to $C$ by $t^i$ in $V_n$ and by $T^\alpha$ in $V_m$. Multiplying both sides of (1.4) by $a_{\alpha\beta}\lambda_3|^\beta$ and summing over $\alpha$, we have

$${}_v\tau_g = a_{\alpha\beta}\lambda_3|^\alpha N_\nu|^\beta{}_{;i}t^i = a_{\alpha\beta}\lambda_3|^\alpha N_\nu|^\beta{}_{,\gamma}T^\gamma,$$

that is,

$${}_v\tau_g = {}_NK_G \cos\theta \tag{2.1}$$

where $\theta$ denotes the angle between $\lambda_3|^\alpha$ and the unit vector $w^\alpha$ defined by

$$N_\nu|^\alpha{}_{,\gamma}T^\gamma = {}_NK_G w^\alpha$$

with ${}_NK_G \geqq 0$.

It is seen from (1.4) that the vector $w^\alpha$ is linearly dependent on the orthogonal vectors $v^\alpha$ and $\lambda_3|^\alpha$. Hence, if ${}_NK_G \neq 0$ and if $\phi$ is the angle between $w^\alpha$ and $v^\alpha$, we have

$$(2.2) \qquad {}_v\tau_g = e_NK_G \sin \phi$$

where $e = \pm 1$.

By the help of the following formula [5, p. 59]

$$\sin \tau = \frac{(\delta^{pq}_{ab} g_{ip} g_{jq} A^a B^b A^i B^j)^{1/2}}{(g_{ij}A^iA^j)^{1/2}(g_{ij}B^iB^j)^{1/2}}$$

where $\tau$ is the angle between two vectors $A^i$ and $B^i$ in a Riemannian space with fundamental covariant tensor $g_{ij}$, we have from (2.2) the following relation

$$(2.3) \qquad ({}_v\tau_g)^2 = \delta^{\lambda\delta}_{\alpha\beta} a_{\lambda\gamma} a_{\delta\tau} N_\nu|^\alpha{}_{;i} N_\nu|^\gamma{}_{;j} v^\beta v^\tau t^i t^j.$$

With the following definitions

$$\Omega_\nu|_{ij} = y^\alpha{}_{;ij} a_{\alpha\beta} N_\nu|^\beta,$$
$$\vartheta_{\mu\nu}|_i = a_{\alpha\beta} N_\nu|^\alpha{}_{;i} N_\mu|^\beta,$$

it is derived that

$$N_\nu|^\alpha{}_{;i} = -\Omega_\nu|_{ik} g^{kj} y^\alpha{}_{,j} + \sum_\mu \vartheta_{\mu\nu}|_i N_\mu|^\alpha.$$

Substituting this expression into (2.3) and simplifying, we have

$$(2.4) \qquad \begin{aligned} ({}_v\tau_g)^2 = {} & g^{hk}\Omega_\nu|_{ik}\Omega_\nu|_{jh} t^i t^j \\ & - \Omega_\nu|_{ik}\Omega_\nu|_{jh} v^k v^h t^i t^j + \delta^{\mu\rho}\vartheta_{\mu\nu}|_i \vartheta_{\rho\nu}|_j t^i t^j. \end{aligned}$$

When $m = n+1$, $V_n$ is a hypersurface of the enveloping space $V_{n+1}$. Denote by $\Omega_{ij}dx^idx^j$ the second fundamental form for $V_n$. Denote by $N^\alpha$ the unit vector normal to $V_n$. Then (2.4) reduces to

$$(2.5) \qquad ({}_v\tau_g)^2 = (g^{hk} - v^h v^k)\Omega_{ik}\Omega_{jh} t^i t^k.$$

If $V_{n+1}$ is a Euclidean space $S_{n+1}$, (2.5) becomes

$$(2.6) \qquad ({}_v\tau_g)^2 = (H_{ij} - \Psi_{ij}) t^i t^j$$

where $H_{ij}$ is the fundamental tensor of the hyperspherical representation of $V_n$ and where $\Psi_{ij}$ is defined by

$$\Psi_{ij} = \Omega_{ik}\Omega_{jh} v^k v^h.$$

When $n=2$, we find from (2.6) by some calculation that for a surface in a Euclidean three dimensional space

$$\begin{aligned}({}_v\tau_g)^2 &= \Omega_{ih}\Omega_{jk}\epsilon^{ah}\epsilon^{bk}t^it^jv^pv^q(g_{ab}g_{pq} - \epsilon_{pa}\epsilon_{qb}) \\ &= g_{ap}g_{bq}\Omega_{ih}\Omega_{jk}\epsilon^{ah}\epsilon^{bk}t^it^jv^pv^q\end{aligned}$$

which implies

$$(2.7) \qquad {}_v\tau_g = e\epsilon^{ah}g_{ap}\Omega_{ih}v^pt^i$$

as obtained in [4].

3. **Theorems.** From (2.1) it is evident that ${}_v\tau_g$ is zero if and only if ${}_NK_G=0$ or $\cos\theta=0$. The first case means that the geodesic curvature of $N_\nu|^\alpha$ along $C$ in $V_m$ is zero; while the second implies that $w^\alpha$ and $v^\alpha$ are coincident, which case arises if $v^\alpha$ is tangent to $C$ and $C$ is a line of curvature of $V_n$ in $V_m$ corresponding to $N_\nu|^\alpha$. Hence we have

THEOREM 3.1. *The indicatric torsion of any unit vector field in $V_n$ along a curve $C$ in $V_n$ relative to a unit vector normal to $V_n$ in $V_m$ is zero if and only if the unit normal vector moves parallelly along the curve in $V_m$ in the sense of Levi-Civita. The indicatric torsion of a unit principal vector field relative to a unit vector normal to $V_n$ in $V_m$ along the corresponding line of curvature of $V_n$ is equal to zero.*

It is shown that in $V_n$ in $S_{n+1}$ the principal curvature of $t^i$ is equal to $(H_{ij}t^it^j)^{1/2}$ and that the normal curvature of $v^i$ with respect to $C$ is equal to $e(\Psi_{ij}t^it^j)^{1/2}$, [3, p. 462; 2, p. 963]. Hence we have from (2.6)

THEOREM 3.2. *The square of the indicatric torsion of a unit vector field along a curve in $V_n$ in $S_{n+1}$ is equal to the difference between the square of the principal curvature of the curve and the square of the normal curvature of the vector field with respect to the curve.*

When $C$ is the curve of $v$, the normal curvature of $v$ with respect to $C$ is the normal curvature of $C$. In this case the above theorem reads as

THEOREM 3.3. *The square of the indicatric torsion of a curve in $V_n$ in $S_{n+1}$ is equal to the difference between the square of its principal curvature and the square of its normal curvature.*

When $C$ is the asymptotic line of $v$, the indicatric torsion along the curve of $v$ of the unit vector field formed by those vectors tangent to $C$ is given from (2.6) by

$$({}_v\tau_g)^2 = H_{ij}v^iv^j.$$

Hence we have

THEOREM 3.4. *The indicatric torsion of the unit asymptotic vector field of a unit vector field along the curve of the latter in $V_n$ in $S_{n+1}$ is numerically equal to the principal curvature of the latter.*

It is obvious from (1.4) that the geodesic torsion $\tau_g$ of a curve on a surface in a Euclidean three dimensional space is equal to the product of the magnitude of $\xi_3$ and sine of the angle between $\xi_1$ and $\xi_3$. Hence we get a short way of deriving the following formula

$$\begin{aligned} \tau_g &= - \epsilon_{ij} d_{kh} g^{hj} t^i t^k \\ &= - \epsilon^{hl} g_{il} d_{kh} t^i t^k. \end{aligned}$$

Let $\eta^i$ denote the unit vectors tangent to the orthogonal trajectories of $C$, that is,

$$\eta^j = e\epsilon^{jh} g_{ih} t^i.$$

The above formula reduces to

$$\tau_g = e d_{kj} t^k \eta^j$$

which gives a new geometric interpretation of the geodesic torsion of a curve as follows:

THEOREM 3.5. *The geodesic torsion of a curve at a point on a surface in a Euclidean three dimensional space is numerically the normal curvature of the curve with respect to its orthogonal trajectory at the point.*

## REFERENCES

1. L. P. Eisenhart, *Riemannian geometry*, Princeton, Princeton University Press, 1949.

2. T. K. Pan, *Normal curvature of a vector field*, Amer. J. Math. vol. 74 (1952) pp. 955–966.

3. ———, *The spherical curvature of a hypersurface in Euclidean space*, Pacific J. Math. vol. 3 (1953) pp. 461–466.

4. ———, *Torsion of a vector field*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 449–457.

5. O. Veblen, *Invariants of quadratic forms*, Cambridge, Cambridge University Press, 1952.

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