

PROJECTIONS ASSOCIATED WITH JACOBI POLYNOMIALS¹

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1. Introduction. The present note is concerned with Jacobi polynomials. However in this section we shall, for the sake of notational simplicity, consider only Legendre polynomials. The statements below, which serve to place our problem in its proper setting are well known and are therefore given without proof.

Let L^2 be the Hilbert space of measurable functions defined on $(-1 \leq x \leq 1)$ with inner product

$$(a, b) = \int_{-1}^1 a(x)\overline{b(x)}dx$$

and let l^2 be the Hilbert space of functions defined on $(n=0, 1, 2, \dots)$ with

$$(\alpha, \beta) = \sum_{n=0}^{\infty} \alpha(n)\overline{\beta(n)}.$$

Let²

$$(1) \quad Da(x) = - [(1 - x^2)a'(x)]',$$

$$(2) \quad \Delta\alpha(n) = \frac{n+1}{2(n+3/2)^{1/2}(n+1/2)^{1/2}} \alpha(n+1) \\ + \frac{n}{2(n+1/2)^{1/2}(n-1/2)^{1/2}} \alpha(n-1).$$

Restricted to the subspace of L^2 consisting of those functions $a(x)$ such that $a(x)$ is bounded and $Da(x) \in L^2$, the operator D is self-adjoint. Its spectrum consists of the points $\lambda = n(n+1)$, $n=0, 1, 2, \dots$, the eigen function corresponding to $n(n+1)$ being $(n+1/2)^{1/2}P_n(x)$ where $P_n(x)$ is the Legendre polynomial of order n . The resolution of the identity G_λ ($-\infty < \lambda < \infty$) corresponding to D , $D = \int \lambda dG_\lambda$, is given by

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² $\alpha(-1)$ is to be interpreted as 0.

$$(3) \quad G_\lambda a(x) = \int_{-1}^1 \left[\sum_{n(n+1) < \lambda} \left(n + \frac{1}{2} \right) P_n(x) P_n(t) \right] a(t) dt.$$

It is possible to consider the projections G_λ in a more general context. Let L^p ($1 \leq p < \infty$) consist of those measurable functions on $(-1 \leq x \leq 1)$ for which $\|a\|_p$ is finite where

$$\|a\|_p = \left[\int_{-1}^1 |a(x)|^p dx \right]^{1/p}.$$

Pollard in some extremely interesting papers [3; 4; and 5] has shown that

$$(4) \quad \|G_\lambda a(x)\|_p \leq A(p) \|a(x)\|_p \quad (4/3 < p < 4)$$

where $A(p)$ depends only upon p and not upon λ . See also Newman and Rudin [2]. The operator Δ is selfadjoint on l^2 . Its spectrum consists of the points $(-1 \leq x \leq 1)$, the formal eigen function corresponding to x being $(n+1/2)^{1/2} P_n(x)$. (Since the spectrum is continuous $(n+1/2)^{1/2} P_n(x)$ does not belong to l^2 as a function of n). If

$$(5) \quad \Gamma_\lambda \alpha(n) = \sum_{m=0}^\infty \alpha(m) \int_{-1}^\lambda \left(n + \frac{1}{2} \right)^{1/2} \left(m + \frac{1}{2} \right)^{1/2} P_m(x) P_n(x) dx$$

then Γ_λ is a resolution of the identity and $\Delta = \int \lambda d\Gamma_\lambda$. Let l^p consist of those functions on $(n=0, 1, 2, \dots)$ for which

$$\|\alpha\|_p = \left(\sum_{n=0}^\infty |\alpha(n)|^p \right)^{1/p}.$$

In the present note we shall show that

$$(6) \quad \|\Gamma_\lambda \alpha(n)\|_p \leq B(p) \|\alpha\|_p \quad (1 < p < \infty),$$

where $B(p)$ depends only upon p and not upon λ . This is of course the analogue of Pollard's theorem.

The Legendre polynomials $P_n(x)$ have as we have noted a double structure: as functions of x they are eigen functions of a second order selfadjoint differential operator and as functions of n they are eigenfunctions of a second order selfadjoint difference operator. It is the interrelation of these structures which makes possible the demonstration of (4) and (6). The initial step in Pollard's demonstration is to simplify (3) using the difference equation satisfied by $P_n(x)$, while here the initial step is to use the differential equation to reduce (5) to a more manageable form.

2. **The main theorem.** Let $\sigma \geq -1/2$, $\tau \geq -1/2$ be fixed. Let

$$\Omega_{\sigma,\tau}(x) = (1-x)^\sigma(1+x)^\tau,$$

$$\begin{aligned} (2n + \sigma + \tau + 1)n! \Gamma(n + \sigma + \tau + 1) h_n^{(\sigma,\tau)} \\ = 2^{\sigma+\tau+1} \Gamma(n + \sigma + 1) \Gamma(n + \tau + 1). \end{aligned}$$

We then have (all formulas such as the one below are taken from [1, vol. 2, Chapter X])

$$\int_{-1}^1 P_n^{(\sigma,\tau)}(x) P_m^{(\sigma,\tau)}(x) \Omega_{\sigma,\tau}(x) dx = \begin{cases} h_m^{(\sigma,\tau)} & m = n, \\ 0 & m \neq n. \end{cases}$$

Let

$$(1) \quad \Gamma_\lambda \alpha(n) = \sum_{m=0}^{\infty} \alpha(m) [h_n^{(\sigma,\tau)} h_m^{(\sigma,\tau)}]^{-1/2} \int_{-1}^{\lambda} P_n^{(\sigma,\tau)}(x) P_m^{(\sigma,\tau)}(x) \Omega_{\sigma,\tau}(x) dx.$$

Note that Γ_λ depends upon σ and τ as well as upon λ .

THEOREM 1. *If $\sigma \geq -1/2$, $\tau \geq -1/2$ then*

$$\|\Gamma_\lambda \alpha(n)\|_p \leq A(\sigma, \tau, \rho) \|\alpha(n)\|_p \quad (1 < p < \infty).$$

In what follows we shall exclude the case $\sigma = -1/2$, $\tau = -1/2$. Our theorem is true for this case but the formulas need slight changes, which we leave to the reader. If $y_n(x) = P_n^{(\sigma,\tau)}(x)$ then

$$[(1-x^2)\Omega(x)y_n'(x)]' + \Omega(x)n(n+\sigma+\tau+1)y_n(x) = 0.$$

Thus

$$\begin{aligned} n(n + \sigma + \tau + 1) \int_{-1}^{\lambda} \Omega(x) y_n(x) y_m(x) dx \\ = - \int_{-1}^{\lambda} y_m(x) [(1-x^2)\Omega(x)y_n'(x)]' dx \\ = - [y_m(x)(1-x^2)\Omega(x)y_n'(x)]_{-1}^{\lambda} + \int_{-1}^{\lambda} y_m'(x)y_n'(x)(1-x^2)\Omega(x) dx, \\ m(m + \sigma + \tau + 1) \int_{-1}^{\lambda} \Omega(x) y_n(x) y_m(x) dx \\ = - \int_{-1}^{\lambda} y_n(x) [(1-x^2)\Omega(x)y_m'(x)]' dx \\ = - [y_n(x)(1-x^2)\Omega(x)y_m'(x)]_{-1}^{\lambda} + \int_{-1}^{\lambda} y_n'(x)y_m'(x)(1-x^2)\Omega(x) dx. \end{aligned}$$

Combining these we obtain for $n \neq m$ the equation

$$\int_{-1}^{\lambda} \Omega(x) y_n(x) y_m(x) dx = \frac{(1 - \lambda^2) \Omega(\lambda) [y_n(\lambda) y_m'(\lambda) - y_m(\lambda) y_n'(\lambda)]}{(n - m)(n + m + \sigma + \tau + 1)}$$

which is the analogue of the Christoffel-Darboux formula. Let

$$\Gamma_1 \alpha(n) = \sum_{m=0, m \neq n}^{\infty} \alpha(m) \frac{(1 - \lambda^2) \Omega(\lambda) y_n(\lambda) y_m'(\lambda) h_n^{-1/2} h_m^{-1/2}}{(n - m)(n + m + \sigma + \tau + 1)},$$

$$\Gamma_2 \alpha(n) = \sum_{m=0, m \neq n}^{\infty} \alpha(m) \frac{(1 - \lambda^2) \Omega(\lambda) y_n'(\lambda) y_m(\lambda) h_n^{-1/2} h_m^{-1/2}}{(n - m)(n + m + \sigma + \tau + 1)},$$

$$\Gamma_3 \alpha(n) = \alpha(n) h_n^{-1} \int_{-1}^{\lambda} [y_n(x)]^2 \Omega(x) dx.$$

Here to simplify the notation we have written Ω for $\Omega_{\sigma, \tau}$ and h_n for $h_n^{(\sigma, \tau)}$. Since $\Gamma_{\lambda} = \Gamma_1 - \Gamma_2 + \Gamma_3$ it is sufficient to show that Γ_1, Γ_2 , and Γ_3 are bounded linear transformations of l^p , the bounds being independent of λ . This is evident for Γ_3 . The following inequality is from Szegő [6, p. 165].

$$| h_n^{-1/2} (1 - \lambda^2)^{1/4} \Omega(\lambda)^{1/2} y_n(\lambda) | \leq M \quad (-1/2 \leq \sigma, -1/2 \leq \tau).$$

Here M depends only upon σ and τ and not upon λ or n . Since

$$2 \frac{d}{dx} P_n^{(\sigma, \tau)}(x) = (n + \sigma + \tau + 1) P_{n-1}^{(\sigma+1, \tau+1)}(x)$$

we find that

$$2 | h_n^{-1/2} (1 - \lambda^2)^{3/4} \Omega(\lambda)^{1/2} y_n'(\lambda) | \leq M(n + \sigma + \tau + 1).$$

Let

$$A(n, \lambda) = h_n^{-1/2} (1 - \lambda^2)^{1/4} \Omega(\lambda)^{1/2} y_n(\lambda),$$

$$B(n, \lambda) = h_n^{-1/2} (1 - \lambda^2)^{3/4} \Omega(\lambda)^{1/2} y_n'(\lambda) \left[n + \frac{1}{2} (\sigma + \tau + 1) \right]^{-1};$$

then

$$| A(n, \lambda) | \leq M_1(\sigma, \tau), \quad | B(n, \lambda) | \leq M_1(\sigma, \tau) \\ (n = 0, 1, 2, \dots; -1 \leq \lambda \leq 1).$$

If we define

$$T_1\alpha(n) = \sum_{m=0, m \neq n}^{\infty} \alpha(m) \frac{m + (\sigma + \tau + 1)/2}{(n - m)(n + m + \sigma + \tau + 1)},$$

$$T_2\alpha(n) = \sum_{m=0, m \neq n}^{\infty} \alpha(m) \frac{n + (\sigma + \tau + 1)/2}{(n - m)(n + m + \sigma + \tau + 1)},$$

then

$$\Gamma_1\alpha(n) = A(n, \lambda)T_1\{\alpha(n)B(n, \lambda)\},$$

$$\Gamma_2\alpha(n) = B(n, \lambda)T_2\{\alpha(n)A(n, \lambda)\},$$

so that it is sufficient to prove that T_1 and T_2 are bounded linear transformations of l^p . Now

$$(T_1 + T_2)\alpha(n) = \sum_{m=0, m \neq n}^{\infty} \alpha(m) \frac{1}{n - m},$$

$$(T_2 - T_1)\alpha(n) = \sum_{m=0, m \neq n}^{\infty} \alpha(m) \frac{1}{n + m + \sigma + \tau + 1}.$$

It is well known that these transformations of l^p are bounded and from this follows the desired result for T_1 and T_2 ; see [7, paragraph 314].

The same methods suffice to prove that if

$$\|\alpha(n)\|_{r,p} = \left[\sum_{n=0}^{\infty} |\alpha(n)|^p (n+1)^{rp} \right]^{1/p}$$

then

$$\|\Gamma_\lambda\alpha\|_{r,p} \leq B\|\alpha\|_{r,p} \quad (1 < p < \infty; -1/p < r < 1 - 1/p).$$

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