ON THE COVERING OF E_n BY SPHERES

DONALD WARNCKE AND FRED SUPNICK

1. Statement of results. Let S_n denote the set of (closed) n-spheres with radii of length $(n^{1/2})/2$ and centers on the lattice points of a rectangular Cartesian coordinate system in E_n (Euclidean n-space). Since $(n^{1/2})/2$ is half the largest diagonal of the unit n-cube, every point of E_n falls on or within some of the spheres of S_n. For $n=1, 2, 3$, if an n-sphere is removed from S_n, then certain points of E_n are not covered by the remaining spheres. However, for $n>3$, proper subsets of S_n cover E_n completely.

Let $[x]$ denote the greatest integer less than or equal to x. We prove the following theorem:

THEOREM. *Each point of E_n is on or within some n-sphere with radius of length $(n^{1/2})/2$ and center at a lattice point $(y_1, \cdots, y_n)$ for which*

$$\sum_{i=1}^{n} y_i \equiv 0 \bmod ([n/4] + 1). \tag{1}$$

2. A lemma. *Let*

1) (x_1, x_2, x_3, x_4) *be any point of* E_4,
2) $\delta_i = x_i - ([x_i]+1/2)$ $(i=1, 2, 3, 4)$,
3) (i_1, i_2, i_3, i_4) *be a rearrangement of* $(1, 2, 3, 4)$ *such that*

$$|\delta_{i_1}| \leq |\delta_{i_2}| \leq |\delta_{i_3}| \leq |\delta_{i_4}|,$$

4) (y_1', y_2', y_3', y_4') *and* $(y_1'', y_2'', y_3'', y_4'')$ *be points in* E_4 *defined as follows*:

(i) $y_1'' = [x_{i_1}]$ *and* $y_1'' = [x_{i_1}]+1$,
(ii) *for* $j=2, 3, 4$

$$y_{i_j}' = y_{i_j}'' = \begin{cases} [x_{i_j}] & \text{if } \delta_{i_j} \leq 0, \\ [x_{i_j}] + 1 & \text{if } \delta_{i_j} > 0. \end{cases}$$

Then

$$\sum_{i=1}^{4} (x_i - y_i')^2 \leq 1 \tag{2}$$

and

Presented to the Society, August 24, 1956; received by the editors May 29, 1956.

$$\sum_{i=1}^{4} (x_i - y_i'')^2 \leq 1. \tag{3}$$

PROOF. Let k range over the set $\{y'_{i_j}, y''_{i_j}\}$ and j over the set $\{2, 3, 4\}$. For each j such that $\delta_{i_j} \leq 0$

$$(x_{i_j} - k) = \left([x_{i_j}] + \frac{1}{2} - |\delta_{i_j}| - [x_{i_j}]\right) = \left(\frac{1}{2} - |\delta_{i_j}|\right). \tag{4}$$

For each j such that $\delta_{i_j} > 0$

$$\begin{aligned}(x_{i_j} - k) &= \left([x_{i_j}] + \frac{1}{2} - |\delta_{i_j}| - ([x_{i_j}] + 1)\right) \\ &= \left(|\delta_{i_j}| - \frac{1}{2}\right).\end{aligned} \tag{5}$$

Therefore

$$\sum_{j=2}^{4} (x_{i_j} - k)^2 = \sum_{j=2}^{4} \left(\frac{1}{2} - |\delta_{i_j}|\right)^2. \tag{6}$$

PROOF OF (2): (i) Suppose $\delta_{i_1} \leq 0$. Then (4) holds if we replace k by y'_{i_j} and all j by 1. Since $|\delta_{i_j}| \leq 1/2$

$$\sum_{j=1}^{4} (x_{i_j} - y'_{i_j})^2 = \sum_{j=1}^{4} \left(\frac{1}{2} - |\delta_{i_j}|\right)^2 \leq 1. \tag{7}$$

(ii) Suppose $\delta_{i_1} > 0$. Then

$$(x_{i_1} - y'_{i_1}) = \left([x_{i_1}] + \frac{1}{2} + |\delta_{i_1}| - [x_{i_1}]\right) = \left(\frac{1}{2} + |\delta_{i_1}|\right). \tag{8}$$

Therefore

$$\begin{aligned}\sum_{j=1}^{4} (x_{i_j} - y'_{i_j})^2 &= \left(\frac{1}{2} + |\delta_{i_1}|\right)^2 + \sum_{j=2}^{4} \left(\frac{1}{2} - |\delta_{i_j}|\right)^2 \\ &\leq \left(\frac{1}{2} + |\delta_{i_2}|\right)^2 + 3\left(\frac{1}{2} - |\delta_{i_2}|\right)^2 \\ &= 1 - 2|\delta_{i_2}| + 4|\delta_{i_2}|^2 \\ &\leq 1\end{aligned} \tag{9}$$

since $|\delta_{i_2}| \leq 1/2$ implies $4|\delta_{i_2}|^2 \leq 2|\delta_{i_2}|$.

PROOF OF (3): (i) Suppose $\delta_{i_1} > 0$. Then (5) holds if we replace k by y''_{i_j} and all j by 1. Therefore (7) holds if we replace y'_{i_j} by y''_{i_j}.

(ii) Suppose $\delta_{i_1} \leq 0$. Then

$$(10)\qquad \begin{aligned}(x_{i_1} - y''_{i_1}) &= \left([x_{i_1}] + \frac{1}{2} - |\delta_{i_1}| - ([x_{i_1}] + 1)\right)\\ &= \left(-\frac{1}{2} - |\delta_{i_1}|\right).\end{aligned}$$

Therefore (9) holds if we replace y'_{i_j} by y''_{i_j}.

It is an immediate consequence of the lemma that the theorem is true for $n=4$:

COROLLARY. *Each point of E_4 is on or within some 4-sphere of unit radius, and center at a lattice point the sum of whose coordinates is even.*

PROOF. By the lemma each point of E_4 is on or within two spheres of unit radius, and centers at lattice points the sum of whose coordinates differ by one. Thus if we remove from S_4 all spheres with centers of odd coordinate sum, the remaining spheres of S_4 still cover E_4.

3. **Proof of the theorem.** Let $n \geq 4$ (if $n=1, 2, 3$ the theorem is obviously true), and

1) $(x_1, \cdots, x_n)$ be a point in E_n,
2) $\delta_i = x_i - ([x_i] + 1/2)$ $(i=1, \cdots, n)$,
3) $\theta_i \equiv \{x_{4i-3}, x_{4i-2}, x_{4i-1}, x_{4i}\}$ $(i=1, \cdots, m)$ where $m=[n/4]$,
4) $\theta_{m+1} \equiv \{x_{4m+1}, x_{4m+2}, \cdots, x_n\}$ (θ_{m+1} having at most three elements).

We shall choose integers $y_1, \cdots, y_n$ satisfying the conditions of the theorem.

Let us consider any θ_i $(1 \leq i \leq m)$. Let (i_1, i_2, i_3, i_4) be a rearrangement of $(4i-3, 4i-2, 4i-1, 4i)$ so that

$$|\delta_{i_1}| \leq |\delta_{i_2}| \leq |\delta_{i_3}| \leq |\delta_{i_4}|.$$

For $j=2, 3, 4$ let

$$(11)\qquad z_{i_j} = \begin{cases}[x_{i_j}] & \text{if } \delta_{i_j} \leq 0,\\ [x_{i_j}] + 1 & \text{if } \delta_{i_j} > 0.\end{cases}$$

By the lemma if we choose z_{i_1} as either $[x_{i_1}]$ or $[x_{i_1}]+1$, then in either case

$$(12)\qquad \sum_{j=1}^{4} (x_{i_j} - z_{i_j})^2 \leq 1.$$

An inequality (12) holds for each θ_i $(i=1, \cdots, m)$. Therefore

$$(13)\qquad \sum_{i=1}^{4m} (x_i - z_i)^2 \leq m.$$

Now let $z_{4m+1}, \cdots, z_n$ be chosen as in (11), i.e. by replacing i_j in (11) by $4m+1, \cdots, n$ successively. Therefore by (4) and (5)

$$\sum_{i=4m+1}^{n} (x_i - z_i)^2 \leq \frac{n - 4m}{4} \cdot \tag{14}$$

Therefore by (13) and (14)

$$\sum_{i=1}^{n} (x_i - z_i)^2 \leq \frac{n}{4} \tag{15}$$

for any $(z_1, \cdots, z_n)$ which may be chosen.

Associated with each point $(x_1, \cdots, x_n)$ of E_n there is a set of 2^m possible lattice points $(z_1, \cdots, z_n)$ which may be selected as in the previous paragraph; let us denote this set by Z. Let $(z_1', \cdots, z_n')$ and $(z_1'', \cdots, z_n'')$ be elements of Z; we shall call them *equivalent* if $\sum_{i=1}^{n} (z_i') = \sum_{i=1}^{n} (z_i'')$. Furthermore, for each element k of the set $\{0, 1, 2, \cdots, m\}$ there exist elements $(z_1^*, \cdots, z_n^*)$ and $(z_1^{**}, \cdots, z_n^{**})$ of Z such that

$$\left| \sum_{i=1}^{n} (z_i^*) - \sum_{i=1}^{n} (z_i^{**}) \right| = k.$$

Therefore Z can be expressed as a sum of mutually exclusive subsets $Z_1, \cdots, Z_{m+1}$ so that (i) any two elements of Z_i are equivalent, and (ii) $n_i = n_{i-1}+1$ $(i=2, \cdots, m+1)$ (where n_i denotes the sum of the coordinates of any element of Z_i). As $n_1, \cdots, n_{m+1}$ are $m+1$ consecutive integers, it must contain one, say n_j, which is divisible by $m+1$. Let $(y_1, \cdots, y_n)$ be any element of Z_j. Then (1) is true. This completes the proof of the theorem.

4. **Some further questions.** It is a corollary of the above proof that there are at least $2^{[n/4]}$ spheres with radii of length $n^{1/2}/2$, and centers at lattice points, which contain $(x_1, \cdots, x_n)$ on or within them. For, (12) holds whether z_{i_1} of Q_i $(i=1, \cdots, [n/4])$ is $[x_{i_1}]$ or $[x_{i_1}]+1$ ($z_{i_2}, z_{i_3}, z_{i_4}$ of Q_i are defined by (11)). (This yields a constructive process for obtaining all $2^{[n/4]}$ of the spheres mentioned above.) Therefore *every* n-sphere (independent of position) with radius of length $n^{1/2}/2$ must contain *at least* $2^{[n/4]}$ lattice points on or within it. The length $n^{1/2}/2$ of the radius is "sharp" with respect to the property of "effective lattice point inclusion" as spheres with radii less than $n^{1/2}/2$ do not necessarily have to contain lattice points on or within them (e.g. those with center at $(1/2, 1/2, \cdots, 1/2)$). The following questions remain open:

I. What is the largest number $N(n)$ so that every n-sphere (independent of position) of radius $n^{1/2}/2$ contains *at least* $N(n)$ lattice points on or within it?

II. Is it possible (for certain n) to replace the number $[n/4]+1$ of the theorem by a number $M(n)$ which is greater than $[n/4]+1$?

BROOKLYN, N. Y. AND
CITY COLLEGE OF THE CITY OF NEW YORK

ON THE MULTIPLICATIVE GROUP OF A DIVISION RING

W. R. SCOTT

Let K be a noncommutative division ring with center Z and multiplicative group K^*. Hua [2; 3] proved that (i) K^*/Z^* is a group without center, and (ii) K^* is not solvable. A generalization (Theorem 1) will be given here which contains as a special case (Theorem 2) the fact that K^*/Z^* has no Abelian normal subgroups. This latter theorem obviously contains both (i) and (ii). As a further corollary it is shown that if M and N are normal subgroups of K^* not contained in Z^*, then $M\cap N$ is not contained in Z^*. The final theorem is that an element x outside Z contains as many conjugates as there are elements in K. This makes more precise a theorem of Herstein [1], who showed that x has an infinite number of conjugates.

Square brackets will denote multiplicative commutation. If S is a set, then $o(S)$ will mean the number of elements in S. A subgroup H of K^* is *subinvariant* in K^* if there is a chain $\{N_i\}$ of subgroups such that $H \lhd N_{r-1} \lhd \cdots \lhd N_1 \lhd K^*$, where $A \lhd B$ means that A is a normal subgroup of B.

LEMMA. *Let K be a division ring, H a nilpotent subinvariant subgroup of K^*, $y\in H$, $x\in K^*$, and $[y, x]=\lambda\in Z^*$, $\lambda\neq 1$. Then the field $Z(x)$ is finite.*

PROOF. The proof of this lemma is essentially part of Hua's proof of (ii), but will be included for the sake of completeness.

Let f be any rational function over Z such that $f(x)\neq 0$. Then

Presented to the Society, November 23, 1956; received by the editors June 15, 1956.