MULTIPLIERS ON COMPLEX HOMOGENEOUS SPACES

R. C. GUNNING[1]

1. Let G be a real Lie group, represented as a transitive group of analytic automorphisms of a simply-connected complex analytic manifold D; if $g \in G$ and $z \in D$, the action of the transformation representing g on the point z will be denoted by gz. A *multiplier* for the group G, with respect to its representation as a transformation group on D, is a C^∞ complex-valued function $\mu(g; z)$ on $G \times D$ which is holomorphic in z and which satisfies $\mu(g_1 g_2; z) = \mu(g_1; g_2 z)\mu(g_2; z)$ for every $g_1, g_2 \in G$; to exclude the obvious trivial case, we further assume that $\mu(g; z) \not\equiv 0$. Such functions are sometimes considered in examining the group G and its representations,[2] but also arise as the continuous analogs of some structures of interest in the study of automorphic functions;[3] our purpose here is to determine the possible multipliers which may arise in connection with the second of the above points of view. We shall always assume here that G is connected. Notice that the set of all multipliers for G forms an abelian group $\mathfrak{M}(G; D)$ under multiplication.

The universal covering group G^* of G also acts as a transformation group on D, the action of the transformation representing $g^* \in G^*$ on the point $z \in D$ being defined by $g^* z = gz$ whenever g^* covers g; we shall consider firstly the group $\mathfrak{M}(G^*; D)$ of multipliers for G^*. Let H^* be the isotropy subgroup of G^* at some point z_0, which point is to be held fixed subsequently, and let K^* be the subgroup of G^* consisting of all elements represented by the trivial transformation which leaves D pointwise fixed. For our purposes, in particular for Siegel's modular groups, there is no loss of generality in assuming: (i) that there are local C^∞ mappings $z \rightarrow g_z^*$ of D into G^* such that $g_z^* z_0 = z$; (ii) that K^* is the center of G^*; (iii) that $K^* \cap [G^*, G^*] = e^*$, that is, the intersection of the center and the commutator subgroup of G^* is the trivial subgroup consisting of the identity e^* alone; and (iv) that elements of finite order are everywhere dense in the group H^*/K^*.

Whenever $f(z)$ is holomorphic and nowhere vanishing on D,

Received by the editors April 27, 1956.

[1] This note was prepared while the author was a National Science Foundation post-doctoral fellow.

[2] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. vol. 48 (1947).

[3] R. C. Gunning, *The structure of factors of automorphy*, Amer. J. Math. vol. 78 (1956).

$\mu_0(g^*; z)=f(g^*z)f(z)^{-1}$ is a multiplier; these are called the trivial multipliers, and form a subgroup $\mathfrak{I}(G^*; D)$ of $\mathfrak{M}(G^*; D)$ which is canonically isomorphic to the group of holomorphic, nowhere-vanishing functions on D. Let $\mathfrak{L}(G^*; D)$ be the additive group of G^*-invariant differential forms of type (0, 1) on the manifold D which are of the form $\bar{\partial}g(z, \bar{z})$ for some C^∞ complex valued function $g(z, \bar{z})$ on D; in the cases which arise from automorphic functions, when D is a Stein manifold, these differential forms are just the $\bar{\partial}$-closed G^*-invariant forms of type (0, 1). Finally let Hom $(K^*; C)$ be the group of C^∞ homomorphisms of K^* into the additive group of complex numbers.

THEOREM. *The group* $\mathfrak{M}(G^*; D)$ *is canonically isomorphic to a direct sum as follows:*

$$\mathfrak{M}(G^*; D) \cong \mathfrak{L}(G^*; D) \oplus \mathfrak{I}(G^*; D) \oplus \text{Hom}\,(K^*; C).$$

PROOF. Since $G^*\times D$ is simply-connected, $\sigma(g^*; z)=\log \mu(g^*; z)$ is a well-defined single-valued function, that branch of the logarithm being selected for which $\sigma(e^*; z)=0$ for the identity $e^*\in G^*$; moreover $\sigma(g_1^*g_2^*; z)=\sigma(g_1^*; g_2^*z)+\sigma(g_2^*; z)$ for every $g_1^*, g_2^*\in G^*$. In particular, whenever $k^*\in K^*$ and $g^*\in G^*$, it follows from assumption (ii) that $g^{*-1}k^*g^*=k^*$, and hence that $\sigma(k^*; z)=\sigma(g^{*-1}k^*g^*; z)=\sigma(g^{*-1}; k^*g^*z)+\sigma(k^*; g^*z)+\sigma(g^*; z)=\sigma(k^*; g^*z)$; therefore $\sigma(k^*; z)=\hat{\sigma}(k^*)$ is a constant. The mapping $k^*\to\hat{\sigma}(k^*)$ is an element of Hom $(K^*; C)$, and the mapping $\mu(g^*; z)\to\hat{\sigma}$ is a homomorphism of $\mathfrak{M}(G^*; D)$ into Hom $(K^*; C)$. Now restricting ourselves to the kernel of the above homomorphism in $\mathfrak{M}(G^*; D)$, we have $\sigma(k^*; z)=0$ for every $k^*\in K^*$. Whenever $h^*\in H^*$ corresponds to an element of finite order in H^*/K^*, say $h^{*n}\in K^*$, then $0=\sigma(h^{*n}; z_0)=n\sigma(h^*; z_0)$; but since such elements are everywhere dense in H^* by assumption (iv), it follows that $\sigma(h^*; z_0)=0$ for every $h^*\in H^*$. Thus for every $g^*\in G^*$, $g^*_{g*z}z_0=g^*z=g^*g_z^*z_0$, so that $g^{*-1}_{g*z}g^*g_z^*\in H^*$; consequently $0=\sigma(g^{*-1}_{g*z}g^*g_z^*; z_0)=-\sigma(g^*_{g*z}; z_0)+\sigma(g^*g_z^*; z_0)$. Now $f(z)=\sigma(g_z^*; z_0)$ is independent of the choice of local sections g_z^*, is clearly a C^∞ function on D by assumption (i), and for any $g^*\in G^*$, $f(g^*z)=\sigma(g^*_{g*z}; z_0)=\sigma(g^*g_z^*; z_0)=\sigma(g^*; z)+f(z)$. Obviously any other function satisfying this functional equation differs from $f(z)$ at most by an additive constant. To each $\sigma(g^*; z)$ associate the differential form $\bar{\partial}f(z)$; this defines a homomorphism of the set of multipliers with $\hat{\sigma}=0$ into the group $\mathfrak{L}(G^*; D)$, and the kernel is clearly precisely the group $\mathfrak{I}(G^*; D)$. To complete the proof, we need merely show that the above homomorphisms are onto. For any differential form $\bar{\partial}f(z)\in\mathfrak{L}(G^*; D)$,

$\mu(g;\, z) = \exp\,(f(gz) - f(z))$ is a multiplier having $\hat{\sigma} = 0$ and mapping onto the form $\bar{\partial} f(z)$ by the previous mapping. Further, any element $\hat{\sigma} \in \mathrm{Hom}\,(K^*;\, C)$ can be extended to an element $\hat{\sigma}_1 \in \mathrm{Hom}\,(G^*;\, C)$; for if we map G^* homomorphically onto the abelianized group $G^*/[G^*, G^*]$, this will be an isomorphism on K^* by assumption (iii), and since a homomorphism on a subgroup of an abelian group can be extended to the full group, $\hat{\sigma}$ clearly admits the desired extension, the exponential of which is a multiplier mapping onto the element $\hat{\sigma}$. This therefore concludes the proof.

Any homomorphic image G_1 of G^* for which the kernel K_1^* of the homomorphism $G^* \to G_1$ is contained in K^* likewise acts as a transformation group on D; the multipliers $\mathfrak{M}(G_1;\, D)$ are determined by the subgroup of $\mathfrak{M}(G^*;\, D)$ consisting of those multipliers $\mu(g^*;\, z)$ for which $\mu(k_1^*;\, z) = 1$ whenever $k_1^* \in K_1^*$.

2. As an example, consider the symplectic group acting on the generalized unit disc of degree p, as introduced by Siegel.[4] The assumptions as listed previously are fulfilled in this case. Moreover the group $\mathfrak{L}(G^*;\, D)$ is a one-dimensional vector space over the complex numbers. To see this, recall that there is but one independent, closed G-invariant differential form of type (1, 1) on D in this case, namely the form Ω determined by the metric.[4] If $\theta_1, \cdots, \theta_a$ form a basis for $\mathfrak{L}(G^*;\, D)$, the elements $\partial\theta_1, \cdots, \partial\theta_a$ must be dependent; hence by a suitable choice of the base, we may assume that $\partial\theta_2 = \cdots = \partial\theta_a = 0$. But then $\theta_j \wedge \bar{\theta}_j$ will be a closed and invariant form for $j \geq 2$, and must be zero since Ω does not admit such a decomposition;[5] hence $\theta_j = 0$ for $j \geq 2$, and $a = 1$. Therefore in this case, the nonobvious multipliers in $\mathfrak{M}(G^*;\, D)$ are powers of the Jacobian determinants of the transformations representing elements of G^*. The same is of course true whenever there is but one closed invariant form of type (1, 1) on D.

UNIVERSITY OF CHICAGO

[4] C. L. Siegel, *Symplectic geometry*, Amer. J. Math. vol. 65 (1943).

[5] This would imply $\Omega \wedge \Omega = 0$, which is impossible for a Kaehler metric.