ON SPACES WITHOUT ISOLATED NONCUT POINTS

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Introduction. Let X denote a topological space and $f: X \to X$ a map. A point $x \in X$ is free if $f(x) \neq x$. f is called a ϕ-map if for every $x \in X$, there exists a free point $x' \neq x$ which does not separate $f(x')$ and x. As a matter of convenience, we allow the identity map as a ϕ-map. Also, let $F(f)$ denote the fixed points of f. If $g: X \to X$ is another map, f and g are said to be F-homotopic provided there exists a homotopy $H: X \times I \to X$ such that $H_0 = f$, $H_1 = g$ and $F(H_t) = F(f)$ for all t, $0 \leq t \leq 1$. In [1], the author proves the following:

THEOREM A. *If X is an ANR (compact metric) and $f: X \to X$ is any ϕ-map, then there exists a map $g: X \to X$, F-homotopic to f, such that $g(X) = X$.*

In this note, we characterize Peano continua X all of whose maps $f: X \to X$ are ϕ-maps as those Peano continua for which the noncut points are dense. Also, we show that for connected ANR's (compact metric) the conclusion of the above Theorem A holds for any map $f: X \to X$ if and only if X fails to possess an isolated noncut point.

We shall assume throughout that all spaces involved are Peano continua and that ANR means ANR (compact metric) (see [2]).

Definitions and preliminaries.

DEFINITION 1. x_0 is an *isolated* noncut point in X if there exists a neighborhood N of x_0 such that $N - x_0$ consists wholly of cut points.

DEFINITION 2. A *tail* in X is a simple arc $\gamma = ab$ such that $\gamma - b$ is open in X. The other end point a is called the *end* of the tail.

DEFINITION 3. An *isolated arc* in X is an arc $\gamma = ab$ such that $\gamma - (a \cup b)$ is open in X and every point of $\gamma - (a \cup b)$ is a cut point of X.

Note that a tail is an isolated arc but not conversely.

The following lemmas are immediate using standard techniques and their proofs are omitted.

LEMMA 1. *A space X possesses an isolated noncut point if and only if X possesses a tail.*

LEMMA 2. *If the noncut points of X are not dense in X, then X possesses an isolated arc.*

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The following lemmas will also be useful in the sequel.

LEMMA 3. *If X has a tail, then there exists a map $f: X \to X$ such that*
(1) $f(X) \subset X$ (*proper inclusion*),
(2) *f is not a ϕ-map,*
(3) *f frees exactly one noncut point,*
(4) *the conclusion of Theorem A fails to hold for f.*

PROOF. Let $\mu: I \to I$ denote the map $\mu(t) = 1/2(|2t-1|+1)$ (μ folds $[0, 1/2]$ onto $[1/2, 1]$). Then μ satisfies the four conditions of the lemma. Now, let $\gamma = ab$ denote a tail in X with a as the end of the tail. Then μ induces a map $f: \gamma \to \gamma$ which can be extended to X by setting $f(x) = x$ on $X - \gamma$. f clearly satisfies the first three conditions of the lemma. To show that the conclusion of Theorem A fails to hold for f we proceed as follows. Let $h: X \to X$ denote any map such that $h(X) = X$ and $F(h) \supseteq F(f)$. Then, by employing the auxiliary map $\bar{h}: \gamma \to \gamma$ defined by $\bar{h}(x) = h(x)$ if $h(x) \in \gamma$ and $\bar{h}(x) = b$ otherwise, we see that $F(h) \cap \gamma = F(\bar{h}) \supset F(f|\gamma)$. Therefore $F(h)$ properly contains $F(f)$ and our lemma follows.

LEMMA 4. *If X has an isolated arc, then there exist maps f_1 and f_2 of X onto X such that*
(1) *f_1 is a ϕ-map which frees no noncut point,*
(2) *f_2 is not a ϕ-map.*

PROOF. (1) Let $\mu: I \to I$ denote any map with the following properties: (a) $\mu(0) = 0$, $\mu(1/2) = 1/2$, $\mu(1) = 1$, (b) $t < \mu(t)$ for $0 < t < 1/2$, (c) $\mu(t) < t$ for $1/2 < t < 1$. Then it is easy to see that such a map μ is a ϕ-map leaving 0 and 1 fixed. Now, let γ denote an isolated arc in X. Then, μ induces a map $f_1: \gamma \to \gamma$ which has the obvious extension to $X - \gamma$, namely $f_1(x) = x$, $x \in X - \gamma$. Then $f_1: X \to X$ is the required map.

(2) Let $\nu: I \to I$ be given by $\nu(t) = t^2$. Then ν is not a ϕ-map and ν leaves 0 and 1 fixed. ν induces $f_2: \gamma \to \gamma$ and is extended to X by using the identity map on $X - \gamma$. $f_2: X \to X$ has then the desired properties.

Results. Given a space X, let $\Omega = X^X$; Ω_0 the "onto" maps of X into X; Φ the class of ϕ-maps in Ω; Φ' the set of maps in Ω which free at least one noncut point; Φ'' the set of maps in Ω which free an infinite number of noncut points. As a matter of convenience, we allow the identity map in the classes Φ, Φ', Φ''. Clearly, $\Phi'' \subseteq \Phi'$ and $\Phi'' \subseteq \Phi$.

THEOREM 1. *The following are necessary and sufficient conditions that X fail to possess a tail:*
(1) $\Phi' = \Phi''$,
(2) $\Omega = \Phi \cup \Omega_0$.

PROOF. (1) The sufficiency is immediate using Lemma 3. To prove the necessity we merely observe that a map which frees a noncut point must free infinitely many since there are no isolated noncut points.

(2) Here again the sufficiency is immediate using Lemma 3. To prove the necessity we proceed as follows: Let $f: X \to X$ be any map. If f frees at least two noncut points, f must be a ϕ-map and hence $f \in \Phi$. The case f frees exactly one noncut point is impossible since we are assuming X has no isolated noncut point. Therefore, it remains to consider the case where f frees no noncut point. In this case we show f is "onto" and hence $f \in \Omega_0$. Deny that $f(X) = X$. Then there exists an $x_0 \notin f(X)$ and x_0 must be a cut point of X. There must exist two noncut points a and b which are separated by x_0, i.e., $X - x_0 = A \mid B$, $a \in A$, $b \in B$. Since $a \in f(X)$ and $b \in f(X)$ the partition of $X - x_0$ induces a partition of $f(X)$ which is impossible since X, and hence $f(X)$, is connected. Therefore, $f \in \Omega_0$ and our proof is complete.

THEOREM 2. *The following are necessary and sufficient conditions that the noncut points of X be dense in X:*

(1) $\Phi = \Phi'$,

(2) $\Omega = \Phi$.

PROOF. (1) *Necessity.* Clearly $\Phi \subseteq \Phi'$, since indeed $\Omega = \Phi'$. Now since the noncut points are dense, $\Phi' = \Phi''$. The general inclusion $\Phi'' \subseteq \Phi$ then gives the desired result, namely $\Phi = \Phi'$.

Sufficiency. If the noncut points were not dense, then X would contain an isolated arc and hence applying Lemma 4 there exists a map $f \in \Phi - \Phi'$ contradicting $\Phi = \Phi'$.

(2) *Necessity.* Obvious.

Sufficiency. If the noncut points were not dense, we may again apply Lemma 4 and obtain a map $f \in \Omega - \Phi$. This contradicts $\Omega = \Phi$ and our proof is complete.

THEOREM 3. *Let X denote an ANR. Then a necessary and sufficient condition that the conclusion of Theorem* A *hold for* **any** *map $f: X \to X$ is that X fail to possess a tail.*

PROOF. The sufficiency follows from (2) of Theorem 1 since Theorem A is trivial for "onto" maps and is valid for ϕ-maps by Theorem A itself. To prove the necessity we merely apply Lemma 3 and our proof is complete.

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SUBSETS OF AN ABSOLUTE RETRACT

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Introduction. In 1939 [**6**] Wojdyslawski asked whether the space $S(X)$ of subsets of an absolute retract X is an absolute retract. This was answered in 1939 [**6**] for the case where X is a bounded closed interval and in 1955 [**2**] for a Peano space. In this note the question is answered for all compact Hausdorff spaces. The author wishes to express her appreciation for the suggestions of Professor W. L. Strother.

Preliminaries. If Y is a space then $S(Y)$ denotes the set of non-null closed subsets of Y with the usual topology [**3**]. (The results of this paper are valid for $S(Y)$ topologized in any other way in which the united extension defined in the following paragraph preserves continuity.) A compact space Y is called a CAR^* if every continuous function from a closed subset M of a normal space N to Y can be extended to a continuous function from N to Y. A definition for M-CAR^* is obtained by replacing functions by multi-valued functions. The following are equivalent [**5**]: (1) X is a CAR^*, (2) X is homeomorphic to a retract of a Tychonoff cube, (3) every cube in which X can be homeomorphically embedded as a closed subset X_0 can be retracted onto X_0.

A multi-valued function $F: X\rightarrow Y$ is said to be continuous at x_0 if (1) V open and $F(x_0)\cap V\neq\emptyset$ implies that there is an open set U containing x_0 such that, for all $x\in U$, $F(x)\cap V\neq\emptyset$ and (2) V open and $F(x_0)\subset V$ implies that there is an open set U containing x_0 such that $F(U)\subset V$. If $f: X\rightarrow Y$ is continuous then $f_s: S(X)\rightarrow S(X)$ defined by $f_s(A)=\bigcup\{f(a)\mid a\in A\}$ is called the **united extension** of f. $S_1(X)$ denotes $S(X)$ and $S_n(X)$ is defined inductively by $S_n(X)=S[S_{n-1}(X)]$.

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