NOTE ON SINGULAR INTEGRALS[1]

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1. **Statement of result.**[2] Let

$$\tilde{f}(x) = (\text{P.V.}/\pi) \int_{-\infty}^{+\infty} [f(y)dy]/(x - y);$$

then by a well-known theorem of M. Riesz,

$$\|\tilde{f}(x)\|_p \leq A_p \|f(x)\|_p, \qquad \text{if } 1 < p < \infty.$$

Hardy and Littlewood [4], and Babenko [1], have complemented this result by proving that:

$$\|\tilde{f}(x) \mid x \mid^\beta\|_p \leq A_{p,\beta} \|f(x) \mid x \mid^\beta\|_p,$$

if $1 < p < \infty$, and $-1/p < \beta < 1/p'$.

The theory of conjugate functions has been extended to n-dimensions by A. P. Calderón and A. Zygmund. In [2] and [3] they have considered a wide variety of singular transformations of the form:

$$T(f)(x) = \text{P.V.} \int_{E_n} \{[H(x, x - y)]/\mid x - y \mid^n\} f(y)dy,$$

and have proved that under suitable conditions on H,

$$\|Tf\|_p \leq A_p \|f\|_p, \qquad 1 < p < \infty.$$

Our aim is to prove the following:

THEOREM. *Let* $(Tf)(x) = P.V. \int_{E_n} \{[H(x, x-y)]/|x-y|^n\} f(y)dy$, *and assume that* $\|T(f)(x)\|_p \leq A_p \|f(x)\|_p$, $1 < p < \infty$. *Assume further that* $|H(x, x-y)| \leq A$. *Then* $\|(Tf)(x)|x|^\beta\|_p \leq A_{p,\beta} \|f(x)|x|^\beta\|_p$ *if* $1 < p < \infty$, *and* $-n/p < \beta < n/p'$.

The proof will depend mainly on the following lemma.

LEMMA. *Let*

$$K(x, y) = \mid 1 - (\mid x \mid / \mid y \mid)^\beta \mid / \mid x - y \mid^n,$$

and let

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[2] The problem considered here was suggested by Professor A. Zygmund.

$$U(f)(x) = \int_{E_n} K(x, y)f(y)dy.$$

Then

$$\|U(f)(x)\|_p \leq A_{p,\beta}\|f(x)\|_p, \qquad \text{if } -n/p < \beta < n/p'.$$

This lemma is well-known if $n=1$, but the proof in the general case is more difficult.

2. **Proof of the lemma.** Given a vector $x=(x_1, x_2, \cdots, x_n)$, let ξ denote the unit vector whose direction is that of x, and $|x| = r$. Similarly let η denote the unit vector whose direction is y and $|y| = R$; that is, $x = r\xi$, and $y = R\eta$. Let $d\omega_\xi$, and $d\omega_\eta$ respectively denote the elements of Euclidean measure on the spheres $|x| = 1$, $|y| = 1$, and let Σ, and Σ' denote these respective unit spheres. In what follows, A will denote a general constant.

Let us set $\lambda = |y|/|x|$. We then notice that the kernel $K(x, y)$ offers three difficulties: near $\lambda=0$, $\lambda=1$, and $\lambda=\infty$. For this reason we break up consideration into these cases. In fact, let $K_1(x, y) = K(x, y)$ if $0 \leq \lambda \leq 1/2$, zero otherwise; $K_2(x, y) = K(x, y)$ if $2 \leq \lambda$, zero otherwise; and $K_3(x, y) = K(x, y)$ if $1/2 < \lambda < 2$, zero otherwise.

Assume, without loss, that $f(x) \geq 0$, and denote by

$$T_i(f) = \int_{E_n} K_i(x, y)f(y)dy, \qquad i = 1, 2, 3.$$

Let us consider T_1 first. Then

$$T_1(f)(r\xi) = \int_{\Sigma'}\int K_1(r\xi, R\eta)f(R\eta)R^{n-1}dRd\omega_\eta \tag{1}$$

now make the change of variables $R = \lambda r$, and (1) becomes:

$$T_1(f)(r\xi) = \int_{\Sigma'}\int_0^{1/2} K_1(\xi, \lambda\eta)f(\lambda r\eta)\lambda^{n-1}d\lambda d\omega_\eta \tag{2}$$

owing to the homogeneity of order $-n$ of $K_1(x, y)$. But we have

$$K_1(\xi, \lambda\eta) = [|1 - \lambda^{-\beta}|]/[1 - 2\lambda \cos(\xi, \eta) + \lambda^2]^{n/2},$$

where (ξ, η) indicates the angle between the vectors ξ and η. While, $[1-2\lambda\cos(\xi, \eta)+\lambda^2]^{n/2} \geq A > 0$, if $0 \leq \lambda \leq 1/2$. Therefore

$$|T_1(f)(r\xi)| \leq A\int_{\Sigma'}\int_0^{1/2} |1 - \lambda^{-\beta}|\lambda^{n-1}f(\lambda r\eta)d\lambda d\omega_\eta. \tag{3}$$

Now define $F_\xi(r)$ by

$$(4)\qquad F_\eta(r) = \int_0^{1/2} |1-\lambda^{-\beta}|\,\lambda^{n-1}f(\lambda r\eta)\,d\lambda$$

and we note that

$$(5)\qquad \left[\int_0^\infty |g(\lambda r)|^p r^{n-1}dr\right]^{1/p} = \lambda^{-n/p}\int_0^\infty |g(r)|^p r^{n-1}dr.$$

Hence applying (5) to (4), via Minkowski's inequality for integrals, gives:

$$(6)\qquad \begin{aligned}&\left[\int_0^\infty |F_\eta(r)|^p r^{n-1}dr\right]^{1/p}\\ &\leq \left[\int_0^{1/2} |1-\lambda^{-\beta}|\cdot\lambda^{n-1}\cdot\lambda^{-n/p}d\lambda\right]\cdot\left[\int_0^\infty |f(r\eta)|^p r^{n-1}dr\right]^{1/p}.\end{aligned}$$

But $\int_0^{1/2}|1-\lambda^{-\beta}|\lambda^{n-1}\cdot\lambda^{-n/p}d\lambda<\infty$ if $\beta<n/p'$. Hence (6) yields

$$(7)\qquad \left[\int_0^\infty |F_\eta(r)|^p r^{n-1}dr\right]^{1/p} \leq A\left[\int_0^\infty |f(r\eta)|^p r^{n-1}dr\right]^{1/p}.$$

Now

$$(8)\qquad \begin{aligned}&\int\left[\int F_\eta(r)d\omega_\eta\right]^p r^{n-1}\,dr \leq A\int\int [F_\eta(r)]^p d\omega_\eta r^{n-1}dr\\ &= A\int d\omega_\eta\int [F_\eta(r)]^p r^{n-1}dr \leq A\int d\omega_\eta\int [f(\eta r)]^p r^{n-1}dr = A\|f\|_p.\end{aligned}$$

Here we have made use of (7) and the fact that the total measure of $d\omega_\eta$ is finite. However by (3)

$$|T_1(f)(r\xi)| \leqq \int_{\Sigma'} F_\eta(r)d\omega_\eta.$$

Therefore (8) implies that

$$\|(T_1)(f)(r\xi)\|_p \leq \left\|\int F_\eta(r)d\omega_\eta\right\|_p \leq A\|f\|_p.$$

The L_p boundedness is therefore proved for $T_1(f)$. The case where $|y|/|x| = \lambda \geq 2$ is treated in the same manner. We get

$$K_2(r\xi, R\eta) = |1-\lambda^{-\beta}|/[1-2\lambda\cos(\xi,\eta)+\lambda^2]^{n/2} \leqq A|1-\lambda^{-\beta}|/\lambda^n.$$

Since $\lambda\geq 2$, matters now depend on the convergence of the integral

$$\int_2^\infty |1-\lambda^{-\beta}|\ [\lambda^{n-1}/\lambda^n]\cdot\lambda^{-n/p}d\lambda,$$

which clearly converges if $\beta > -n/p$.

We now proceed to the case where $1/2 < \lambda < 2$, and in the previous notation we have:

$$T_3(f)(r\xi) = \int_{\Sigma'} d\omega_\eta \int_{1/2}^2 \{[|1-\lambda^{-\beta}|\cdot\lambda^{n-1}]/[1-2\lambda\cos(\xi,\eta)+\lambda^2]^{n/2}\} \cdot f(\lambda r\eta)d\lambda \tag{9}$$

and write $G_\lambda(r\xi)$ for the following:

$$G_\lambda(r\xi) = \int_{\Sigma'} f(\lambda r\eta)\{[1-\lambda^{-\beta}|\cdot\lambda^{n-1}]/[1-2\lambda\cos(\xi,\eta)+\lambda^2]^{n/2}\}d\omega_\eta. \tag{10}$$

Now observe that if $1/2 < \lambda < 2$, the expression within the brackets is bounded up to a constant multiple by Poisson's Kernel for the $(n-1)$ sphere. Using a well known property of the Poisson Kernel, we get from (10)

$$\int_\Sigma |G_\lambda(r\xi)|^p d\omega_\xi \le A\int_{\Sigma'} |f(\lambda r\eta)|^p d\omega_\eta, \qquad \text{if } 1/2 < \lambda < 2. \tag{11}$$

Now integrate (11) with respect to $r^{n-1}dr$; this gives

$$\int_0^\infty \int_\Sigma |G_\lambda(r\xi)|^p d\omega_\xi r^{n-1}dr \le A\int_0^\infty \int_{\Sigma'} |f(\lambda r\eta)|^p d\omega_\eta r^{n-1}dr. \tag{12}$$

However $\|f(\lambda x)\|_p = \lambda^{-n/p}\|f(x)\|_p$. Therefore (12) yields

$$\|G_\lambda\|_p \le A\lambda^{-n/p}\|f\|_p. \tag{13}$$

However,

$$T_3(f) = \int_{1/2}^2 G_\lambda d\lambda. \tag{14}$$

Therefore, Minkowski's inequality for integrals applied to (13) and (14) gives

$$\|T_3(f)\|_p = \left\| \int_{1/2}^2 G_\eta d\lambda \right\| \le \int_{1/2}^2 \|G_\lambda\|_p d\lambda \le A\|f\|_p$$

and the proof of the lemma is complete.

3. Proof of the theorem. Let now

$$T(f) = \text{P.V.} \int_{E_n} \{[H(x, x-y)]/|x-y|^n\} f(y)dy,$$

where T satisfies the conditions of the theorem. Write $F(x) = T[f(y)]$, and $F^*(x) = T[|y|^\beta f(y)]$. Then by the assumptions of the theorem:

$$\|F^*(x)\|_p \leq A_p \| |x|^\beta f(x)\|_p.$$

It is therefore sufficient to prove that:

$$\|F^*(x) - |x|^\beta F(x)\|_p \leq A_{p,\beta} \| |x|^\beta f(x)\|_p. \tag{15}$$

However,

$$\begin{aligned} &|F^*(x) - |x|^\beta F(x)| \\ &= \left| \int_{E_n} \{[H(x, x-y)]/|x-y|^n\} [|y|^\beta - |x|^\beta] f(y)dy \right| \\ &= \left| \int_{E_n} \{[H(x, x-y)]/|x-y|^n\} [1 - (|x|^\beta/|y|^\beta)] |y|^\beta f(y)dy \right| \\ &\leq A \int_{E_n} K(x, y) |y|^\beta |f(y)| dy, \end{aligned}$$

by the assumption that $|H(x, x-y)| \leq A$.

Hence we need only appeal to the lemma to complete the proof.

References

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