BOUNDS FOR DETERMINANTS. II[1]

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1. In this article there are derived, for the determinants of certain matrices of (real or complex) constants, lower and upper bounds which are suitable for use with automatic computing machinery.

Let $A=(a_{ij})_{1\cdots n}$ be an arbitrary matrix; let m_i be the maximum modulus of the elements of the ith row: $m_i=\max_j |a_{ij}|$. A lower bound for det A is given by the relation

$$(1)\qquad |\det A| \geqq \left[1-\sum_1^n m_j(|a_{jj}|+m_j)^{-1}\right]\prod_1^n(|a_{ii}|+m_i).$$

A routine can be so programmed that the first factor is computed first; if this factor is negative or 0, the routine skips the rest of the program. If the first factor is positive, this lower bound is useful especially because of the ease with which the elements of the matrix can be scaled.

It is to be remarked that the bounds of this note apply without formal change not only to the given matrix, but to any other matrix (c_{ij}) for which the relations $|c_{ii}|=|a_{ii}|$, $\max_j |c_{ij}|\leqq m_i$ hold. A single computation gives a bound for a family of matrices.

2. The following abbreviations are used to simplify the typography.

$$(2)\qquad a'_{ij}=a_{ij}-a_{i1}a_{1j}/a_{11}\qquad\qquad (i,j>1);\ A'=(a'_{ij}).$$

$$(3)\qquad m_i=\max_{j\neq 1}|a_{ij}|.$$

$$(4)\qquad T_j=m_j(m_j+|a_{jj}|)^{-1},\qquad 0(0^{-1})=1.$$

$$(5)\qquad f_i=1-\sum_1^{i-1}T_j,\qquad f_1=1,\qquad f'_i=1-\sum_2^{i-1}T'_j,\qquad f'_2=1.$$

$$(6)\qquad S=\sum_1^n T_i,\qquad S'=\sum_2^n T'_i.$$

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(7) $\quad G_i = m_i[T_i^{-1} - f_i^{-1}].$

Moreover, m_i', T_j', G_i' are defined by priming (3), (4), (7) respectively. The bound (1) needs to be established only under the hypothesis $S<1$. The first step is

LEMMA 1. *If $S<1$, then $S'<1$.*

PROOF. Note $a_{11}\neq 0$ if $S<1$. From (2), (3) the relations

$$m_i' \leqq m_i(1 + m_1/|a_{11}|), \qquad |a_{ii}'| \geqq |a_{ii}| - m_1m_i/|a_{11}|$$

follow. Hence the following relations are true.

$$\begin{aligned} T_i' &= 1 - \frac{|a_{ii}'|}{m_i' + |a_{ii}'|} \leqq 1 - \frac{|a_{ii}'|}{m_i(1 + m_1/|a_{11}|) + |a_{ii}'|} \\ &= \frac{m_i(1 + m_1/|a_{11}|)}{m_i(1 + m_1/|a_{11}|) + |a_{ii}'|} \qquad (8) \\ &\leqq (1 + m_1/|a_{11}|)T_i = (1 - T_1)^{-1}T_i. \end{aligned}$$

By use of this estimate for T_i together with the hypothesis $S<1$ in the form $\sum_2^n T_i<1-T_1$, the conclusion

$$S' = \sum_2^n T_i \leqq (1 - T_1)^{-1} \sum_2^n T_i < 1$$

follows.

COROLLARY (Ostrowski [1]). *Let m_i, T_i, S be defined by* (3), (4), (6). *The hypothesis $S<1$ is enough to establish the conclusion* $\det A\neq 0$.

In view of the relation

(9) $$\det A = a_{11} \det A',$$

this is an obvious consequence of Lemma 1.

In attending to the proof of (1), we note first that the right member of (1) can be rewritten $\prod_1^n G_i=|a_{11}|\cdot\prod_2^n G_i$, and proceed to establish the relations $G_i'\geqq G_i$. The induction hypothesis $|\det A'| \geqq \prod_2^n G_i'$ will finish the argument by (9).

LEMMA 2. *$G_i'\geqq G_i$ $(i>1)$.*

PROOF. The explicit assumption $\sum_1^{i-1} T_j<1$ is made. Without it, (1) is trivial.

We use the estimate (8). Thus we obtain

$$f_i' = 1 - \sum_2^{i-1} T_j' \geqq 1 - (1 - T_1)^{-1} \sum_2^{i-1} T_j = (1 - T_1)^{-1} f_i;$$

$$f_i'^{-1} \leqq (1 - T_1) f_i^{-1}.$$

Further, as shown above, $m_i' \leqq (1 - T_1)^{-1} m_i$. Next, since $f_i'^{-1} - 1$ is positive, we can obtain

$$\begin{aligned} m_i' - m_i' f_i'^{-1} &= - m_i' (f_i'^{-1} - 1) \geqq - m_i (1 - T_1)^{-1} [(1 - T_1) f_i^{-1} - 1] \\ &= m_i (1 - T_1)^{-1} - m_i f_i^{-1}. \end{aligned}$$

Thus we can reason as follows

$$\begin{aligned} G_i' &= m_i' [T_i'^{-1} - f_i'^{-1}] \geqq |a_{ii}'| + m_i (1 - T_1)^{-1} - m_i f_i^{-1} \\ &\geqq |a_{ii}| - m_i m_1 / |a_{11}| + m_i (1 - T_1)^{-1} - m_i f_i^{-1} \\ &= G_i. \end{aligned}$$

Using similar arguments, the following upper bound for $|\det A|$ can be obtained. It is valid if the hypothesis $S < 1$ is satisfied.

$$|\det A| \leqq \prod_1^n H_i, \quad H_i = |a_{ii}| + m_i (f_i^{-1} - 1).$$

Reference

1. A. M. Ostrowski, *Sur les conditions générales pour la régularité des matrices*, Rendiconti di Mat. e delle sue Appl. vol. 10 (1951) pp. 156–168.

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