A THEOREM ON POWER SERIES WHOSE COEFFICIENTS HAVE GIVEN SIGNS

W. H. J. FUCHS[1]

1. The following theorem, first proved by A. Hurwitz and G. Pólya, is well known ([**3**] or [**1**, p. 99]).

If $\sum_{k=0}^{\infty} a_k z^k$ *is a power series of finite radius of convergence, then it is possible to find a sequence* $\{\epsilon_k\}$ $(\epsilon_k = \pm 1)$ *such that the series* $\sum_{k=0}^{\infty} \epsilon_k a_k z^k$ *has the circle of convergence as natural boundary.*

In this note I prove the following companion-piece to Pólya's theorem.

THEOREM. *If* $\{\epsilon_k\}_{k=0}^{\infty}$ *is a sequence with* $\epsilon_k = \pm 1$, *then there is always a power series* $\sum a_k z^k$, $a_k > 0$, *of finite radius of convergence such that the series* $\sum \epsilon_k a_k z^k$ *can be analytically continued across a semi-circle on its circle of convergence.*

This theorem answers in the negative the question: Is there a "universal scrambling sequence" $\{\epsilon_k\}$, $\epsilon_k = \pm 1$, turning every power series $\sum a_k z^k$ with positive coefficients into a power series $\sum \epsilon_k a_k z^k$ having the circle of convergence as natural boundary? This problem was raised by Mrs. Turán, and I am indebted to Dr. P. Erdös for communicating it to me.

An example (§4) shows that the semi-circle in the statement of the theorem can not be replaced by a larger arc.

A question which remains open is to find a corresponding theorem for the case that $\{\epsilon_k\}$ is a given sequence of complex numbers of absolute value one.

2. The following lemmas are required.

LEMMA 1. *Let* $\Lambda = \{\lambda_n\}$ *be a sequence of positive numbers no two of which are at a distance less than* $c > 0$ *from each other. Let*

$$g(z) = \prod_{\lambda \in \Lambda} \frac{\lambda - z}{\lambda + z} e^{2z/\lambda}. \tag{1}$$

Then there are constants A *and* B *such that in* $x \geqq 0$, $|z - \lambda| \geqq c/4 (\lambda \in \Lambda)$

$$0 < (Be^{\phi(r)})^x \leqq |g(z)| \leqq (Ae^{\phi(r)})^x$$

where $r = |z| = |x + iy|$ *and*

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$$\phi(r) = \sum_{\lambda<r;\lambda\in\Lambda} 2/\lambda.$$

For a proof of this lemma see [2, Lemmas 3 and 4].

LEMMA 2. *Let $M=\{\mu\}$ be a sequence of positive numbers whose mutual distances are $\geqq 1$. Suppose that the function $h(\zeta)=h(\xi+i\eta)$ is regular in the region $\xi\geqq 0$, $\zeta\neq 0$ except for simple poles at the points $\zeta=\mu\in M$. Suppose further that there are positive constants A, α, β $(\beta<\pi)$ such that*

$$|h(\xi)| = |h(\xi+i\eta)| < Ae^{\alpha\xi-\beta|\eta|}$$

in $\xi\geqq 0$, $\zeta\neq 0$, except in circles of radius $1/4$ with centers at the points $\mu\in M$.

Then the function

$$H(z) = \sum_{\mu\in M} r_\mu z^\mu$$

is regular in the sector $0<|z|$, $|\arg z|<\beta$; where r_μ is the residue of $h(\zeta)$ at μ.

PROOF. Let C_R be the semicircle $|\zeta|=R$, $\xi\geqq 0$ and let L_R be a curve with endpoints $\zeta=iR$ and $\zeta=-iR$ which runs along the imaginary axis except for an indentation into the right half plane near $\zeta=0$.

By the residue theorem

$$\frac{1}{2\pi i}\int h(\zeta)e^{-k\zeta}d\zeta = \sum_{\mu<R;\mu\in M} r_\mu e^{-k\mu}$$

where the integration is along C_R+L_R. If the number k is chosen positive and larger than α, then on C_R

$$|h(\zeta)e^{-k\zeta}| < Ae^{(\alpha-k)\xi-\beta|\eta|},$$

and therefore

$$\int_{C_R} h(\zeta)e^{-k\zeta}d\zeta \to 0$$

as $R\to\infty$ through a sequence of values avoiding the intervals $|\xi-\mu|<1/4$ on the real axis. It follows that for $z=e^{-k}$ $(k>\alpha)$ the series

$$\sum_{\mu\in M} r_\mu z^\mu$$

converges to a function

$$(2) \qquad H(z) = \sum r_\mu z^\mu = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\zeta) z^\zeta d\zeta,$$

where the path of integration is the imaginary axis with an indentation near $\zeta = 0$. For purely imaginary values of ζ, $|z^\zeta| = |z^{i\eta}| = e^{-\eta\cdot\arg z}$. Hence, in $|\arg z| \leqq \beta' < \beta$, $|h(i\eta)z^{i\eta}| < Ae^{-\beta|\eta|+|\eta|\cdot|\arg z|} < Ae^{-(\beta-\beta')|\eta|}$. This shows that the integral on the right-hand side of (2) is uniformly convergent in $|\arg z| \leqq \beta' < \beta$. Therefore it defines the analytic continuation of $H(z)$ into the whole sector $|\arg z| < \beta$.

3. Proof of the theorem. Let $\{\lambda\} = \Lambda$ be the set of those odd multiples of $1/2$ for which $\epsilon_{\lambda-1/2}\epsilon_{\lambda+1/2} = -1$. Write

$$\phi(r) = \sum_{\lambda<r} 2/\lambda,$$

$$m = \liminf_{r\to\infty} (\phi(r) - \log r),$$

$$M = \limsup_{r\to\infty} (\phi(r) - \log r).$$

We consider separately the five cases:

(i) $-\infty < M < \infty$. (ii) $M = -\infty$. (iii) $-\infty < m < \infty$. (iv) $m = \infty$. (v) $m = -\infty$, $M = \infty$.

These cases are not mutually exclusive, but they cover all possibilities.

(i) $-\infty < M < \infty$. Define $g(z)$ by (1). By Lemma 1

$$0 < \limsup_{n\to\infty} |g(n)|^{1/n}/n = C < \infty \quad (n = 1, 2, \cdots).$$

The function

$$h(\zeta) = (C\zeta)^{-\zeta} g(\zeta) \operatorname{cosec} \pi\zeta$$

satisfies the hypotheses of Lemma 2 with $\beta = \pi/2$, $M = \{1, 2, 3, \cdots\}$. The residue of $h(\zeta)$ at n is

$$r_n = (-1)^n g(n)/\pi(Cn)^n.$$

By the choice of C, the series

$$\sum r_n(-z)^n = \sum c_n z^n = f(z)$$

has radius of convergence 1. The sign of c_n is the same as that of $g(n)$. But $g(x)$ changes sign between those integers k, $k+1$ for which ϵ_k and ϵ_{k+1} are of opposite sign and nowhere else. Hence $\epsilon_n g(n)$ is of constant sign. Also, by Lemma 2, $f(z)$ is regular in $|\arg(-z)| < \pi/2$, i.e. in $x<0$. Therefore one of the two functions $\pm f(z)$ has the required properties.

(ii) $M=-\infty$. We can find a sequence $\{\nu\}$ of odd multiples of $1/2$ which has no terms in common with Λ and for which $\lim\sup\{\phi(r)+\sum_{\nu<r}4/\nu-\log r\}=0$, say. The construction of the previous case can now be used, if $g(\zeta)$ is replaced by

$$g(\zeta)\cdot\left\{\Pi\frac{\nu-\zeta}{\nu+\zeta}e^{2\zeta/\nu}\right\}^2.$$

(iii) $-\infty<m<\infty$. Let $g(\zeta)$ again be defined by (1). If D is any positive number, the function

$$h(\zeta)=(D\zeta)^{\zeta}/g(\zeta)$$

satisfies the hypotheses of Lemma 2, with $\{\mu\}=\{\lambda\}$, $\beta=\pi/2$.[2] The residue at $\lambda=\rho$ is

$$r_\rho=(D\rho)^\rho/g'(\rho).$$

Now

$$g'(\rho)=-\prod_{\lambda\in\Lambda;\lambda\neq\rho}\frac{\lambda-\rho}{\lambda+\rho}e^{2\lambda/\rho}\cdot\frac{e^2}{2\rho}$$

and so, by Lemma 1 $g'(\rho)$ lies between

$$(B_1e^{\phi(\rho)})^\rho \quad\text{and}\quad (A_1e^{\phi(\rho)})^\rho,$$

where A_1 and B_1 are independent of ρ. Since $\lim\inf(\phi(\rho)-\log\rho)$ is finite, the constant D can be adjusted so that the series

$$\sum_{\rho\in\Lambda}r_\rho\cdot z^{\rho-1/2}=\psi(z)$$

has radius of convergence 1. By Lemma 2 $\psi(z)$ is regular in $x>0$. The values of ϵ at successive terms of the sequence $\{\rho-1/2\}$ are of opposite sign, since two such integers are separated by exactly one term of the sequence Λ.

The coefficients r_ρ of two consecutive terms in the power series are also of opposite sign, since the slope of $g(x)$ has opposite signs at successive zeros ρ of $g(x)$.

Therefore $\pm\psi(z)$ satisfies all requirements, except that it has zero coefficients for all integers which are not of the form $\rho-1/2$, $\rho\in\Lambda$. By adding to $\pm\psi(z)$ an entire function whose power series has suitable signs, an example satisfying all requirements is obtained.

[2] This is easily verified, if the inequality $\theta\sin\theta\leqq(\pi/2)(1-\cos\theta)$ is noted.

(iv) $m=\infty$. We can choose a sub-sequence of Λ such that we have case (iii) for the new sequence and such that any two consecutive terms of the sub-sequence are separated by an even number of terms in the original sequence Λ. The construction of case 3 can now be applied.

(v) $m=-\infty$, $M=\infty$. Put

$$\gamma(r) = \phi(r) - \log r = \sum_{\lambda<r} 2/\lambda - \log r.$$

The hypothesis $m=-\infty$, $M=\infty$ implies that given $q>0$ there are arbitrarily large numbers h, k such that

$$\inf_{h<r<k} \gamma(r) - \gamma(h) = -q \tag{3}$$

and

$$0 \leqq \gamma(k) - \gamma(h) \leqq 1. \tag{4}$$

A set $M=\{\mu\}$ is now constructed as follows.

First a sequence of nonoverlapping intervals $I_1, I_2, \cdots$ is chosen such that in $I_q=(h_q, k_q)$ (3) and (4) hold, with $h=h_q$ and $k=k_q$, and such that $h_{q+1}>10k_q$ $(q=1, 2, \cdots)$. All terms of Λ which lie in I_q are terms of the new sequence M, in fact $M\cap I_q=\Lambda\cap I_q$ $(q=1, 2, \cdots)$.

If $\delta(r)=\sum_{\mu<r;\mu\in M} 2/\mu-\log r$, then it follows that (3) and (4) hold with δ in place of γ and $h=h_q$, $k=k_q$.

Next we select further positive odd multiples of 1/2 as members of M.

At first enough of these are chosen from the interval $(0, h_1)$ to make

$$0 < \delta(h_1) \leqq 1.$$

This gives, by (4),

$$0 \leqq \delta(k_1) \leqq 2.$$

Next enough terms are added between k_1 and h_2 to make

$$0 \leqq \delta(r) \leqq 2 \qquad (k_1 \leqq r \leqq h_2)$$

and

$$1 \leqq \delta(h_2) \leqq 2.$$

Then, by (3)

$$1 \leqq \delta(k_2) \leqq 3$$

and we make

$$1 \leqq \delta(r) \leqq 3 \qquad (k_2 \leqq r \leqq h_3),$$

$$2 \leqq \delta(h_3) \leqq 3.$$

Continuing in this way we can choose M so that

$$q - 2 \leqq \delta(r) \leqq q \qquad (k_{q-1} \leqq r \leqq h_q;\ q = 2, 3, \cdots).$$

Hence $\delta(r)\to\infty$ as $r\to\infty$ through any set of values outside the intervals I_q. But by (3)

$$-1 \leqq \liminf \delta(r) \leqq 0.$$

Since $\delta(r)$ has local minima at the $r\in M$, there is a $\mu=\mu_q^*\in I_q$ such that

$$-1 \leqq \delta(\mu_q^*) \leqq 0; \qquad \liminf \delta(r) = \liminf \delta(\mu_q^*).$$

Since

$$\delta(h_q) - \delta(\mu_q^*) = \log \mu_q^*/h_q - \sum_{h_q\leqq\mu<\mu_q^*} 2/\mu \geqq q - 2$$

$$\mu_q^* > 3h_q \qquad (q \geqq 4). \tag{5}$$

Similarly

$$\mu_q^* < k_q/3 \qquad (q \geqq 4). \tag{6}$$

Now let $g_1(z)$ be defined by (1) with M in place of Λ. As in case (iii) we derive a function

$$\psi(z) = \sum_{\mu\in M} r_\mu z^{\mu-1/2}$$

from the auxiliary function $h(\zeta)=(D\zeta)^\zeta/g(\zeta)$. This function $\psi(z)$ has the following properties:

1. Its power series has radius of convergence 1.
2. As $\mu\to\infty$ through the sequence $\mu_1^*, \mu_2^*, \cdots,$

$$\limsup \left| r_\mu \right|^{1/(\mu-1/2)} = 1.$$

3. $\psi(z)$ is regular in $x>0$.
4. Consecutive terms of the series have opposite signs.
5. The sign of $\epsilon_\mu r_\mu$ is the same for all μ from one and the same interval I_q.

All these properties are proved as in case 3.

Now let n_q be the largest integer not exceeding $\mu_q^*/2$. For any choice of signs

$$\chi(z) = \sum_{q=4}^{\infty} \pm\ (z(1+z)^2/4)^{n_q} = \sum c_n z^n$$

is regular in the domain $|z(1+z)^2|<4$ which contains $|z|\leqq 1,\ z\neq 1$. The coefficient c_n is 0, whenever n is not in one of the intervals

$n_q \leqq n \leqq 3n_q$. By (5) and (6) this implies that $c_n = 0$ for $n \notin I_q$ $(q=1, 2, \cdots)$. All the nonzero c_n whose indices are in a fixed interval I_q have the same sign. As $n \to \infty$ through the sequence $\mu_1^*, \mu_2^*, \cdots$, $|c_n|^{1/n} \to 1$. This shows that $\chi(z)$ has radius of convergence 1 and so $z=1$ is the only singularity of $\chi(z)$ on the unit circle. The function formed by Hadamard multiplication of $\chi(z)$ and $\psi(z)$, i.e. $\sum_{\mu \in M} r_\mu c_{\mu-1/2} z^{\mu-1/2} = f(z)$ is therefore analytic at all points of $|z|=1$ at which ψ is analytic, i.e. on $|z|=1, x>0$. If the $\pm$ signs in the definition of $\chi(z)$ are chosen suitably, it follows from property 5 of $\psi(z)$ that the coefficient of $z^{\mu-1/2}$ in $f(z)$ and $\epsilon_{\mu-1/2}$ are of the same sign, since the power series of $f(z)$ contains no terms whose indices are not in one of the intervals I_q. By adding an entire function to $f(z)$, we can form a power series without vanishing terms which satisfies all requirements.

4. The following example shows that there are sequences $\{\epsilon_k\}$ such that every closed semi-circle on the circle of convergence of $\sum \epsilon_k a_k z^k$ $(a_k > 0)$ contains at least one singularity.

Let

$$\epsilon_k = 1, \qquad k \equiv 0, 1 \pmod 4,$$
$$\epsilon_k = -1, \qquad k \equiv 2, 3 \pmod 4.$$

Then $F(z) = \sum \epsilon_k a_k z^k$ $(a_k > 0, \limsup a_k^{1/k} = 1)$ is a power series whose sequence of coefficients has sign-changes with density 1/2. By a theorem of Pólya (see [1, p. 51]) this implies that $F(z)$ has a singularity on $|z|=1$, $|\arg z| \leqq \pi/2$. But $F(-z)$ is again a function whose power series coefficients have sign-changes of density 1/2. Therefore Pólya's theorem shows that $F(z)$ has a singularity on $|z|=1$, $|\arg(-z)| \leqq \pi/2$. Since F has real coefficients, the singularities of F are symmetrically situated with respect to the real axis. It is now easy to see that F has singularities on every closed semi-circle on $|z|=1$.

References

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Cornell University